Definitions: 1. A subset $\{u_\alpha\}_{\alpha \in A} \subset \mathcal{H}$ is orthonormal if

i) $\|u_\alpha\| = 1$ for all $\alpha \in A$

ii) $u_\alpha \perp u_\beta$ whenever $\alpha \neq \beta$ $\quad \langle u_\alpha, u_\beta \rangle = 0$

2. $\{x_n\}_{n=1}^{\infty}$ l.i in $\mathcal{H}$ there is a standard process Gram-Schmidt converting $\{x_n\}$ into an orthonormal sequence $\{u_n\}$ s.t. $\operatorname{span}\{x_n\}_{n=1}^{N} = \operatorname{span}\{u_n\}_{n=1}^{N}$ (*) $\forall N$

$$u_1 = \frac{x_1}{\|x_1\|} \qquad u_2 = \frac{x_2 - \langle x_2, u_1 \rangle u_1}{\|x_2 - \langle x_2, u_1 \rangle u_1\|} \qquad \langle u_1, u_2 \rangle = 0$$

Assume $x_1 \dots x_{N-1}$ o.n $\{$ s.t (*) holds.

$$u_N = \frac{\tilde{u}_N}{\|\tilde{u}_N\|}. \qquad \tilde{u}_N = x_N - \sum_{i=1}^{N-1} \langle x_N, u_i \rangle u_i$$

$$\operatorname{span}\{x_n\}_{n=1}^{N} = \operatorname{span}\{u_n\}_{n=1}^{N-1}$$

$$\langle u_N, u_j \rangle = 0 \qquad j = 1, \dots, N-1$$

Bessel's inequality: If $\{u_\alpha\}_{\alpha \in A}$ is an orthonormal set in $\mathcal{H}$ for any $x \in \mathcal{H}$

$$\sum_{\alpha \in A} |\langle x, u_\alpha \rangle|^2 \leq \|x\|^2$$

in particular $\{\alpha : \langle x, u_\alpha \rangle \neq 0\}$ at most countable.

Pf: If suffices to show for any $F \subset A$ finite

$$\sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2 \leq \|x\|^2$$

$$0 \leq \Big\| x - \sum_{\alpha \in F} \langle x, u_\alpha \rangle u_\alpha \Big\|^2 = \|x\|^2 - 2\mathrm{Re} \Big\langle x, \sum_{\alpha \in F} \langle x, u_\alpha \rangle u_\alpha \Big\rangle$$

$$+ \Big\| \sum_{\alpha \in F} \langle x, u_\alpha \rangle u_\alpha \Big\|^2 \quad \text{(Pythagorean theo)}$$

$$= \|x\|^2 - 2\mathrm{Re} \sum_{\alpha \in F} \overline{\langle x, u_\alpha \rangle} \langle x, u_\alpha \rangle + \sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2$$

$$= \|x\|^2 - \sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2 \geq 0$$

Theorem: If $\{u_\alpha\}_{\alpha \in A}$ is an orthonormal set in $\mathcal{H}$, t.f.a.e

a) Completeness: if $\langle x, u_\alpha \rangle = 0 \quad \forall \alpha \Rightarrow x = 0$

b) Parseval's identity $\|x\|^2 = \sum_{\alpha \in A} |\langle x, u_\alpha \rangle|^2 \quad \forall x \in \mathcal{H}$

c) For $x \in \mathcal{H}$ $x = \sum_{\alpha \in A} \langle x, u_\alpha \rangle u_\alpha$ where the sum on the right has only at most countably non-zero terms and converges (in the norm topology) no matter how these terms are ordered.

Definition: An orthonormal set having the properties a) - c) is called an orthonormal basis.

Examples: $L^2[0, 2\pi] \cap \{u(0) = u(2\pi)\}$ $\quad \langle u_n, u_j \rangle = 0$, $\langle v_n, v_i \rangle = 0$

$\tilde{u}_n = \frac{1}{\sqrt{2}} u_n$ / $u_n = \cos nx$ $\quad n = 0, 1, \ldots$ orthogonal $\quad \langle u_i, v_j \rangle = 0$

$\tilde{v}_n = \frac{1}{\sqrt{2}} v_n$ | $v_n = \sin nx$ $\quad n = 1, \ldots$ $\quad \frac{1}{\pi} = \int_0^{2\pi} \cos^2\theta \, d\theta = \frac{1}{2} 2\pi$

ℓ^2, $\quad u_n = (0, \ldots, 0, \overset{n\text{-position}}{1}, \ldots, 0)$

$C[0, 2\pi]$ $\quad \{e^{inx}\}_{n=0}^{\infty}$ (not example)

$\uparrow$

not Hilbert space

$(C[0, 2\pi], \| \quad \|_u)$ is not Hilbert, because it is not given by inner product.

$(C[0, 2\pi] : \| \quad \|_2)$ is not complete.

a) $\Rightarrow$ c) By Bessel $\sum_{\alpha \in A} |\langle x, u_\alpha \rangle|^2 \leq \|x\|^2$

countably many non-zero terms $\alpha_1 \dots \alpha_n \dots$ enumeration of α's s.t $\langle x, u_\alpha \rangle \neq 0$

By Bessel's inequality $\sum_{j=1}^{\infty} |\langle x, u_{\alpha_j} \rangle|^2 < \infty$ (*)

↑ converges

We want $\| x - \sum_{j=1}^{n} \langle x, u_{\alpha_j} \rangle u_{\alpha_j} \| \longrightarrow 0$ as $n \to \infty$

① $\sum_{j=1}^{\infty} \langle x, u_{\alpha_j} \rangle u_{\alpha_j}$ converges

$S_n = \sum_{j=1}^{n} \langle x, u_{\alpha_j} \rangle u_{\alpha_j}$

⇑ Cauchy

$$\left\| \sum_{j=n}^{n+k} \langle x, u_{\alpha_j} \rangle u_{\alpha_j} \right\|^2 = \sum_{j=n}^{n+k} |\langle x, u_{\alpha_j} \rangle|^2 \longrightarrow_{n,k \to \infty} 0$$

$\{S_n\}$ converges as $\mathcal{H}$ is complete

② if converges to x, $y = x - \sum_{j=1}^{\infty} \langle x, u_{\alpha_j} \rangle u_{\alpha_j}$

$$\langle y, u_\alpha \rangle = \langle x, u_\alpha \rangle - \sum_{j=1}^{\infty} \langle x, u_{\alpha_j} \rangle \underbrace{\langle u_{\alpha_j}, u_\alpha \rangle}_{\delta_{\alpha \alpha_j}}$$

$\alpha \neq \alpha_j \ \forall j$ ⇒ $\langle x, u_\alpha \rangle = 0$

$= 0 \quad \forall \alpha \longrightarrow y = 0$ (a)

$\alpha \neq \alpha_j \quad \forall j \quad \Rightarrow \quad \langle x, u_\alpha \rangle = 0$

$$\langle y, u_\alpha \rangle = \underbrace{\langle x, u_\alpha \rangle}_{0} - \sum_{j=1}^{\infty} \langle x, u_{\alpha_j} \rangle \underbrace{\langle u_{\alpha_j}, u_\alpha \rangle}_{0} = 0$$

$\alpha = \alpha_i$

$$\langle y, u_{\alpha_i} \rangle = \langle x, u_{\alpha_i} \rangle - \sum_{j=1}^{\infty} \langle u, u_{\alpha_j} \rangle \underbrace{\langle u_{\alpha_j}, u_{\alpha_i} \rangle}_{\substack{1 \ i=j \\ 0 \ i \neq j}}$$

$$= \langle x, u_{\alpha_i} \rangle - \langle x, u_{\alpha_i} \rangle = 0$$

c) $\Rightarrow$ b) $\qquad x = \sum_{\alpha \in A} \langle x, u_\alpha \rangle u_\alpha$ use notation above.

$$0 \leq \|x\|^2 - \sum_{j=1}^{n} |\langle x, u_{\alpha_j} \rangle|^2 = \underbrace{\left\| x - \sum_{j=1}^{n} \langle x, u_{\alpha_j} \rangle u_{\alpha_j} \right\|^2}_{\downarrow \ n \to \infty \ 0}$$

$$\Downarrow$$

$$\|x\|^2 = \sum_{j=1}^{\infty} |\langle x, u_{\alpha_j} \rangle|$$

Proposition: Every Hilbert space has an orthonormal basis.

Proposition: A Hilbert space $\mathcal{H}$ is separable iff it has a countable orthonormal basis, in which case every orthonormal basis for $\mathcal{H}$ is countable.