Definition: $(\mathcal{H}, \|\ \|)$ is a Hilbert space if $\mathcal{H}$ is complete with the norm $\|\ \|$ given by the inner product.

Proposition if $x_n \longrightarrow x$ & $y_n \longrightarrow y$ then ($\{\|y_n\|\}$ is bounded)

$$\langle x_n, y_n \rangle \longrightarrow \langle x, y \rangle$$

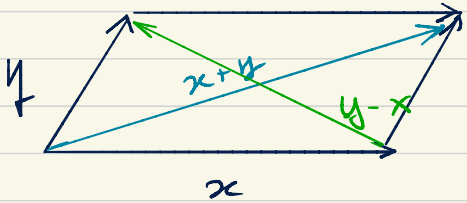
$$|\langle x_n, y_n \rangle - \langle x, y \rangle| = |\langle x_n - x, y_n \rangle + \underbrace{\langle x, y_n \rangle - \langle x, y \rangle}_{\langle x,\, y_n - y \rangle}|$$

$$\leq |\langle x_n - x, y_n \rangle| + |\langle x, y_n - y \rangle|$$

$$\leq \underbrace{\|x_n - x\|}_{n \to \infty,\ L \to 0} \underbrace{\|y_n\|}_{< M} + \|x\| \underbrace{\|y_n - y\|}_{\downarrow 0,\ n \to \infty} \longrightarrow 0 \qquad n \to \infty$$

Parallelogram law: For all $x, y \in \mathcal{H}$

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

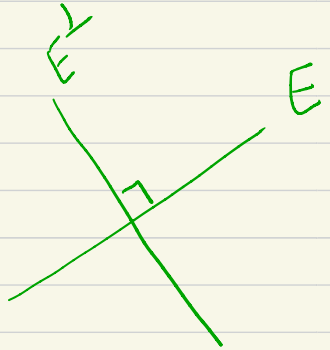


$$\|x \pm y\|^2 = \|x\|^2 \pm 2\,\mathrm{Re}\,\langle x, y \rangle + \|y\|^2$$

$$\|x+y\|^2 + \|y-x\|^2 = 2\|x\|^2 + 2\|y\|^2$$

Definition: If $x, y \in \mathcal{H}$ we say that x is orthogonal to y $x \perp y$ if $\langle x, y \rangle = 0$. If $E \subset \mathcal{H}$

$$E^{\perp} = \{ x \in \mathcal{H} : \langle x, y \rangle = 0 \quad \forall y \in E \}$$

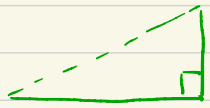


Remark: $E^{\perp}$ is a closed subspace of $\mathcal{H}$.

$x_1, x_2 \in E^{\perp}$ $\quad \langle \underbrace{a_1 x_1 + a_2 x_2}_{\in E^{\perp}}, y \rangle = a_1 \underbrace{\langle x_1, y \rangle}_{0} + a_2 \underbrace{\langle x_2, y \rangle}_{0}$ $\quad \forall y \in E$

The Pythagorean Theorem: If $x_1, \dots x_n \in \mathcal{H}$ & $x_i \perp x_j$ $i \neq j$ then

$$\| \sum_{j=1}^{n} x_j \|^2 = \sum_{j=1}^{n} \| x_j \|^2$$



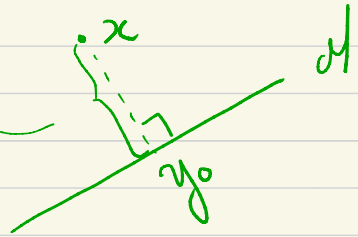
$$\| \sum_{j=1}^{n} x_j \|^2 = \langle \sum_{j=1}^{n} x_j , \sum_{k=1}^{n} x_k \rangle = \sum_{j,k=1}^{n} \langle x_j, x_k \rangle$$

$$= \sum_{j=1}^{n} \| x_j \|^2$$

Theorem: Suppose $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, and $x \in \mathcal{H}$. Then

(i) there exists a unique element $y_0 \in \mathcal{M}$ which is the closest to x in the sense that

$$\|x - y_0\| = \inf_{y \in \mathcal{M}} \|x - y\|$$



ii) the element $z_0 = x - y_0$ is orthogonal to $\mathcal{M}$ i.e

$$\langle x - y_0, y \rangle = 0 \qquad \forall y \in \mathcal{M}.$$

iii) thus $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$, i.e each $x \in \mathcal{H}$ can be expressed uniquely as $x = y_0 + z_0$ with $y_0 \in \mathcal{M}$ and $z_0 \in \mathcal{M}^{\perp}$

Pf: $x \in \mathcal{H}$ $\quad \delta = \inf_{y \in \mathcal{M}} \{\|x - y\|\}$ $\quad x \notin \mathcal{M}$

$y_n \in \mathcal{M}$ $\quad \|x - y_n\| \to \delta$ $\quad$ Goal: show $\{y_n\}$ converges

$\|x - y_n\| \longrightarrow \delta \quad n \to \infty \qquad y_n \in M \qquad (\delta = \inf_{y \in M} \|y - x\|)$

Claim : $\{y_n\}$ is Cauchy ($\Leftarrow$ M closed $\Rightarrow$ M complete then $y_n \to y_0 \in M$)

$\hookrightarrow \|x - y_n\| \to \|x - y_0\| = \delta$

Parallelogram law

$$2\|y_n - x\|^2 + 2\|y_m - x\|^2 = \|y_n - y_m\|^2 + \|y_n + y_m - 2x\|^2$$

$$= \|y_n - y_m\|^2 + 4\left\|\frac{y_n + y_m}{2} - x\right\|^2$$

$$\|y_n - y_m\|^2 = 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - 4\left\|\underbrace{\frac{y_n + y_m}{2}}_{\in M} - x\right\|^2$$

$$\left\|\frac{y_n + y_m}{2} - x\right\| \geq \delta$$

$$\|y_n - y_m\|^2 \leq \underbrace{2\|y_n - x\|^2}_{\downarrow\, n \to \infty} + \underbrace{2\|y_m - x\|^2}_{\downarrow\, m \to \infty} - 4\delta^2 \longrightarrow 0$$

$2\delta^2 \qquad 2\delta^2 \qquad m, n \to \infty$

$z_0 = x - y_0$ $\qquad f(t) = \|z_0 + tu\|^2 \geq 0$ $\qquad u \in M$ $\qquad \langle z_0, u \rangle \in \mathbb{R}$

$$f(t) = \|z_0\|^2 + 2t\langle z_0, u\rangle + t^2\|u\|^2$$

$$\left(f(t) = \|x - y_0 + tu\|^2 = \|x - \underbrace{(y_0 - tu)}_{\in M}\|^2 \right.$$

$$f(t) \geq \delta^2 = \|z_0\|^2$$

f has a minimum at $t = 0$

$$f'(t)\Big|_{t=0} = 2\langle z_0, u\rangle = 0$$

$$\langle x - y_0, u\rangle = 0$$

$$x = y_0 + (x - y_0) = \underset{\in M}{y_0} + \underset{\in M^\perp}{z_0}$$

assume

$$x = y_0' + z_0' \implies \underbrace{y_0 - y_0'}_{\in M} = \underbrace{z_0' - z_0}_{\in M^\perp}$$

$y_0' \in M$, $z_0' \in M^\perp$

$$0 = \langle y_0 - y_0', z_0' - z_0\rangle = \langle y_0 - y_0', y_0 - y_0'\rangle = \|y_0 - y_0'\|^2 \implies y_0 = y_0'$$

$$z_0 = z_0'$$

Let $\mathcal{H}$ be a Hilbert space. If $y \in \mathcal{H}$ $f_y : \mathcal{H} \to \mathbb{C}(\mathbb{R})$ defined by $f_y(x) = \langle x, y \rangle$ is a bounded linear functional

$$|f_y(x)| = |\langle x, y \rangle| \leq \|x\| \|y\| \quad \text{by Cauchy-Schwarz}$$

$$\Rightarrow \quad \|f_y\| \leq \|y\| \quad \text{since } f_y(y) = \|y\|^2 \text{ then}$$

$$\|f_y\| = \|y\|$$

Theorem (Riesz Representation)

If $f \in \mathcal{H}^*$ there is a unique $y \in \mathcal{H}$ such that

$f(x) = \langle x, y \rangle \quad \forall x \in \mathcal{H}$ — Moreover $\|f\| = \|y\|$

Pf if $f \equiv 0 \Rightarrow y = 0$ / $f \not\equiv 0$

$\mathcal{M} = \{ x \in \mathcal{H} : f(x) = 0 \}$ closed linear subspace $\mathcal{M}^\perp \neq \{0\}$

For $z \in \mathcal{M}^\perp$ $\|z\| = 1$, $x \in \mathcal{H}$

$z \in \mathcal{A}^\perp \qquad \|z\| = 1 \qquad x \in \mathcal{H} \qquad u = f(x)z - f(z)x$

$$f(u) = f(x)f(z) - f(z)f(x) = 0 \Rightarrow u \in \mathcal{A}$$

$$\langle u, z \rangle = 0 \qquad \langle f(x)z - f(z)x ; z \rangle = 0$$

$$f(x)\langle z, z \rangle - f(z)\langle x, z \rangle = 0$$

$$f(x)\|z\|^2 = f(z)\langle x, z \rangle$$

$$f(x) = \langle x, \overline{f(z)}\, z \rangle \qquad y = \overline{f(z)}\, z$$

uniqueness $\qquad \langle x, y_1 \rangle = \langle x, y_2 \rangle = f(x) \qquad \forall x \in \mathcal{H}$

$$\langle x ; y_1 - y_2 \rangle = 0 \qquad \forall x \in \mathcal{H} \qquad x = y_1 - y_2$$

$$\|y_1 - y_2\|^2 = 0 \Rightarrow y_1 = y_2$$

Remark: Hilbert spaces are reflexive $\qquad \mathcal{H} \sim \mathcal{H}^* \sim (\mathcal{H}^*)^*$