The Open mapping theorem

Let X, Y be Banach spaces. If $T \in \mathcal{L}(X, Y)$ is surjective then it is open.

Proof: Enough to show that $\overline{TB_1^X} = \overline{TB_1}$ contains a ball centered at 0, $B_n = B^X(0, n)$

$$X = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} nB_1, \quad T \text{ surjective} \quad Y = TX = \bigcup_{n=1}^{\infty} n\,\overline{TB_1}$$

Y complete $\Rightarrow$ $\overline{TB_1}$ is not nowhere dense

Baire category

$\exists\, y_0 \in \operatorname{int} \overline{TB_1}$ $\exists r > 0$ $B^Y(y_0, 4r) \subset \overline{TB_1}$ &

Claim: $B^Y(0, r/2) \subset \overline{TB_1}$

If $\|y\| < r/2m \Rightarrow y \in \overline{TB_{1/2m}} \Rightarrow B(0, r) \subset \overline{TB_1}$

Goal: show that if $\|y\| < r/2$, $\exists z \in B_1$ s.t. $y = Tz$

$\|y\| < r/2m \Rightarrow y \in \overline{TB_{1/2m}}$

Since $y \in B(0, r/2) \Rightarrow y \in \overline{TB_{1/2}}$, $\exists z_1 \in B_{1/2}$

$\|y - Tz_1\| < r/4$ (bec $B(y, r/4) \cap TB_{1/2} \neq \phi$)

$\Rightarrow y - Tz_1 \in \overline{TB_{1/4}}$ $\exists z_2 \in B_{1/4}$ s.t

$\|y - Tz_1 - Tz_1\| < r/8$ (bec $B(y - Tz_1, \frac{r}{8}) \cap TB_{1/4} \neq \phi$

$\vdots$

$\exists z_n \in B_{1/2^n}$; $\|z_n\| < \frac{1}{2^n}$ $\|y - \sum_{j=1}^{n} Tz_j\| < \frac{r}{2^{n+1}}$

$\Rightarrow y - \sum_{j=1}^{n} Tz_j \in \overline{TB_{1/2^{n+1}}}$ — (1)

Note $\sum_{n=1}^{\infty} \|z_n\| < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty \Rightarrow \sum_{n=1}^{\infty} z_n = z \in X$ (X complete)

$\|z\| \leq \sum_{n=1}^{\infty} \|z_n\| < 1 \Rightarrow z \in B_1$ (*)

(*) $\sum_{n=1}^{N} z_n \longrightarrow z$ as $N \to \infty$

$T\left(\sum_{n=1}^{N} z_n\right) \longrightarrow Tz$ as $N \to \infty$ bec $T \in \mathcal{L}(X, Y)$

by (1) $\left\| y - T\left(\sum_{n=1}^{N} z_n\right) \right\| \leq \frac{r}{2^{N+1}}$

$N \to \infty$ $\quad y = Tz$

The Closed graph theorem

Let X, Y be Banach spaces & $T: X \to Y$ is a closed linear map then T is continuous.

Pf $\pi_1 : X \times Y \to X$, $(x,y) \to x$, $\pi_2 : X \times Y \to Y$, $(x,y) \to y$ open maps

$\pi_1(x, Tx) = x$ $\quad$ $\pi_2(x, Tx) = Tx$

$\pi_1 \in \mathcal{L}(\text{graph } T, X)$

since π_1 is surjective (π_1 is 1-1) $\Rightarrow$ π_1 is open

$\pi_2 \in \mathcal{L}(\text{graph } T, Y)$ $\quad$ $\Downarrow$ π_1^{-1} is continuous

$\text{graph } T = \{(x, Tx) : x \in X\}$

closed subspace in $X \times Y$

in Banach is Banach

Goal write T as a comp. of continuous maps

$$X \xrightarrow{\pi_1^{-1}} \text{graph } T \xrightarrow{\pi_2} Y$$

$$x \to (x, Tx)$$

$$T = \pi_2 \circ \pi_1^{-1}$$

Hilbert spaces

Let $\mathcal{H}$ be a complex vector space. An inner product (or scalar product) on $\mathcal{H}$ is a map $\langle \, , \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$ s.t.

i) $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle \quad \forall x, y, z \in \mathcal{H} \quad \forall a, b \in \mathbb{C}$

ii) $\langle y, x \rangle = \overline{\langle x, y \rangle}$

iii) $\langle x, x \rangle \geq 0 \quad \forall x \in \mathcal{H}$ & $\langle x, x \rangle > 0$ if $x \neq 0$

Remarks: i) & ii) $\langle x, ay + bz \rangle = \bar{a} \langle x, y \rangle + \bar{b} \langle x, z \rangle$

real inner product $\mathbb{C}$ is $\mathbb{R}$ then ii) $\rightsquigarrow$ ii') $\langle x, y \rangle = \langle y, x \rangle$

$\| x \| = \sqrt{\langle x, x \rangle}$

Define: $\|x\| = \sqrt{\langle x, x\rangle}$.

Cauchy - Schwarz inequality: $|\langle x, y\rangle| \le \|x\|\,\|y\| \quad \forall x, y \in \mathcal{H}$
with equality iff x & y are linearly dependent.

Pf: $\langle x, y\rangle = 0$ ✓ $\qquad \langle x, y\rangle \neq 0 \qquad \|x\| = \|y\| = 1, \; t \in \mathbb{R}$

$$0 \le \|x - ty\|^2 = f(t) = \|x\|^2 + t^2\|y\|^2 - t\langle x, y\rangle - t\langle y, x\rangle$$

$$= \|x\|^2 - t\left[\langle x, y\rangle + \overline{\langle x, y\rangle}\right] + t^2\|y\|^2$$

$$= \|x\|^2 - 2t\,\mathrm{Re}\,\langle x, y\rangle + t^2\|y\|^2 \ge 0$$

$$\Delta = 4\left[(\mathrm{Re}\,\langle x, y\rangle)^2 - \|x\|^2\|y\|^2\right] \le 0$$

$$(\mathrm{Re}\,\langle x, y\rangle)^2 \le \|x\|^2\|y\|^2 \Rightarrow |\mathrm{Re}\,\langle x, y\rangle| \le \|x\|\,\|y\| \quad (*)$$

$\langle x, y\rangle \neq 0 \qquad \langle x, y\rangle = e^{i\alpha}|\langle x, y\rangle| \Rightarrow \langle x, e^{i\alpha}y\rangle = |\langle x, y\rangle|$, $\in \mathbb{R}$

apply $(*)$ to x & $e^{i\alpha}y$ (rather than y)

$$|\mathrm{Re}\,\langle x, e^{i\alpha}y\rangle| = |\langle x, y\rangle| \le \|x\|\,\|e^{i\alpha}y\| = \|x\|\,\|y\|.$$

Proposition : The function $x \longmapsto \|x\|$ is a norm

Pf $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}\,\langle x, y\rangle$

$\leq \|x\|^2 + \|y\|^2 + 2\,|\mathrm{Re}\,\langle x, y\rangle|$

$\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|$ ↓ CS

$\leq (\|x\| + \|y\|)^2$

Definition: $(\mathcal{H}, \|\ \|)$ is a Hilbert space if $\mathcal{H}$ is complete with the norm $\|\ \|$ given by the inner product.

Examples: $(X, \mathcal{A}, \mu)$ $\quad L^2(\mu) = \{ f : X \to \mathbb{C} \ : \ \int |f|^2 \, d\mu < \infty \}$

$$\langle f, g \rangle = \int f \bar{g} \, d\mu \qquad \text{since} \qquad |f\bar{g}| \leq \frac{1}{2}|f|^2 + \frac{1}{2}|g|^2$$

Hilbert space, $\quad X = \mathbb{N} \quad \mathcal{A} = \mathcal{P}(\mathbb{N}) \quad \mu =$ counting measure

$$\ell^2 = \{ (a_j)_{j=1}^{\infty} \ : \ \sum |a_j|^2 < \infty \}$$

$\mathcal{H}$ Hilbert space