Definition: Let X, Y be topological (metric) spaces $f: X \to Y$ is said to be open if $f(U)$ is open in Y whenever U is open in X.

Example: X, Y normed vector space $\|(x,y)\| = \|x\| + \|y\|$

$\pi: X \times Y \to X$

$(x,y) \mapsto \pi(x,y) = x$ is open

Remark: Let X, Y be normed vector spaces $f: X \to Y$ linear

f is open iff $\exists r > 0$ such that

$$B^Y(0,r) \subset f(B^X(0,1))$$

$\Rightarrow$ ✓

$\Leftarrow$ $U \subset X$ open $y \in f(U)$ $\exists x \in U$ $f(x) = y$

since U open $\exists \delta > 0$ $B^X(x,\delta) \subset U \Rightarrow f(B^X(x,\delta)) \subset f(U)$

$$f(B^X(x,\delta)) = f(x + \delta B^X(0,1)) = f(x) + \delta f(B^X(0,1))$$

$$f(B^X(x,\delta)) = f(x) + \delta f(B^X(0,1)) \subset f(U)$$

$\exists r > 0 \qquad B^Y(0,r) \subset f(B^X(0,1))$

$$\underbrace{f(x) + \delta B^Y(0,r)}_{B^Y(f(x),\delta r)} \subset f(x) + \delta f(B^X(0,1)) \subset f(U)$$

$B^Y(f(x),\delta r) \subset f(U) \Rightarrow y = f(x) \in \inf f(U)$

$\Rightarrow f(U)$ open

The Open mapping theorem

Let X, Y be Banach spaces. If $T \in \mathcal{L}(X, Y)$ is surjective then it is open.

Corollary: If X, Y Banach spaces & $T \in \mathcal{L}(X, Y)$ is bijective, then T is an isomorphism (ie T^{-1} bijective)

Pf: T surjective $\Rightarrow$ T open

T injective $\Rightarrow$ T^{-1} exists $\Rightarrow$ T^{-1} is continuous

$T^{-1} : Y \longrightarrow X$

$(T^{-1})^{-1}(U) = T(U)$ open

$\uparrow$ open in X $\quad$ T is open

Corollary: Suppose that V is a Banach space w.r.t 2 norms $\| \ \|_1$ & $\| \ \|_2$. Assume there is $C > 0$ such that $\forall v \in V$ $\|v\|_1 \leq C \|v\|_2$ then $\| \ \|_1$ & $\| \ \|_2$ are equivalent.

Pf: $Id : (V, \| \ \|_2) \longrightarrow (V, \| \ \|_1)$ $\quad Id(x) = x$ is a bounded linear map

Id is bijective : $Id : (V, \| \ \|_1) \longrightarrow (V, \| \ \|_2)$ is bounded

X, Y Banach $\quad T \in \mathcal{L}(X,Y)$ surjective $\Rightarrow$ T open

- enough to show $\overline{TB_1^X} = \overline{TB_1}$ contains a ball centered at 0 $\quad (B^Y(0,\delta) \subset \overline{TB_1}$ want)

- $X = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} nB_1 \qquad B_n = B^X(0,n)$

T surjective $\quad Y = TX = \bigcup_{n=1}^{\infty} n\overline{TB_1}$, Y complete (Baire category)

there is n s.t $n\overline{TB_1}$ is not nowhere dense, thus

$\overline{TB_1}$ is not nowhere dense $\quad \operatorname{int}\overline{TB_1} \neq \emptyset \quad \exists y_0 \in \operatorname{int}\overline{TB_1}$

$\exists r > 0$

$$B^Y(y_0, 4r) \subset \overline{TB_1} \Rightarrow \begin{cases} \exists y_1 \in B^Y(y_0, 2r) \cap TB_1 \quad (y_0 \in \overline{TB_1}) \\ \exists x_1 \in B_1 \\ y_1 = Tx_1 \end{cases}$$

Claim: $B^Y(0, r/2) \subset TB_1$

Step 1 $\quad B^Y(0,2r) \subset \overline{TB_1}$ $\quad B(y_1, 2r) \subset B(y_0, 4r) \subset \overline{TB_1}$

$\|y\| < 2r \qquad y = y + y_1 - Tx_1 \subset -Tx_1 + B(y_1, 2r) \subset -Tx_1 + \overline{TB_1}$

$\overline{T(-x_1 + B_1)}$

$\|y\| < 2r$ $\quad y \in \overline{T(-x_1 + B_1)}$ $\quad x_1 \in B_1 \quad -x_1 \in B_1$

$-x_1 + B_1 \subset 2B_1$

$y \in \overline{T(2B_1)} = 2\overline{TB_1}$

$y/2 \in \overline{TB_1} \Rightarrow \boxed{B(0,r) \subset \overline{TB_1}}$

$n \in \mathbb{N} \quad y \in Y \quad \|y\| < r/2^n \Rightarrow \|2^n y\| < r$

$2^n y \in B(0,r) \subset \overline{TB_1} \Rightarrow y \in \overline{TB_{1/2^n}}$

$$\boxed{\|y\| < r/2^n \Rightarrow y \in \overline{TB_{1/2^n}}}$$

We want $y \in B(0, r/2) \quad \exists z \in B_1 \quad y = Tz$

Definitions: ① X, Y normed vector spaces $T: X \to Y$ linear

the graph of T

$$\text{graph } T = \{(x, Tx) : x \in X\} \subset X \times Y$$

② $T: X \to Y$ linear is closed if graph T is a closed subset of $X \times Y$.

Remark: If $T: X \to Y$ continuous then T is closed.

$(x_n, y_n) \in$ graph T $\quad (x_n, y_n) \to (x, y)$ $\quad T$ continuous.

$x_n \to x \Rightarrow Tx_n \to Tx = y$ graph T is closed

The Closed graph theorem

Let X, Y be Banach spaces & $T: X \to Y$ is a closed linear map then T is continuous.

Question: Why is the closed graph theorem useful?

(1) To show continuity in general: assume $x_n \to x$

a) Show $\{Tx_n\}$ converges

b) $Tx_n \to Tx$

(2) To show continuity via closed graph theorem

$$\begin{cases} x_n \to x \\ Tx_n \to y \end{cases} \Rightarrow \underbrace{y = Tx}$$