Baire category theorem & Applications

If X topological space / metric space $E \subset X$ is **no-where dense** if

$$\operatorname{int} \bar{E} = \emptyset \quad \text{or} \quad (\bar{E})^c \text{ is dense}$$

Definitions : X topological / metric space

i) $E \subset X$ is **first category** or **meager** in X if E is a countable union of nowhere dense sets

ii) If E is not of first category in X then it is said to be of **second category**

iii) The complement of a meager set is a **generic** or **residual** set.

Topology of complete metric spaces

Baire category theorem : Let X be a complete metric space

1) If $\{U_n\}_{n=1}^{\infty}$ is a sequence of open dense subsets of X then

$$\bigcap_{n=1}^{\infty} U_n \text{ is dense in } X$$

2) X is NOT a countable union of nowhere dense sets.

Pf : W open, goal $W \cap \bigcap_{n=1}^{\infty} U_n \neq \emptyset$

Since W is open & U_1 is dense $\exists x_1 \in U_1 \cap W$, since U_1 is also open $W \cap U_1$ is open $\exists r_1 \in (0, 1/2)$ s.t

$$\overline{B(x_1, r_1)} \subset B(x_1, 2r_1) \subset W \cap U_1$$

Since U_2 is dense $B(x_1, r_1) \cap U_2 \neq \emptyset$; $\exists x_2 \in B(x_1, r_1) \cap U_2$

($B(x_1, r_1)$ open set)

$\exists r_2 \in (0, \frac{1}{2^2})$ $\overline{B(x_2, r_2)} \subset B(x_1, r_1) \cap U_2 \subset W \cap U_1 \cap U_2$

Assume that x_j & $r_j \in (0, \frac{1}{2^j})$ have been chosen for $j < n$

s.t. $j \geq 1$ $\quad \overline{B(x_j, r_j)} \subset B(x_{j-1}, r_j) \cap U_j$

Since U_n is dense $\quad \underbrace{B(x_{n-1}, r_{n-1})}_{\text{open}} \cap U_n \ni x_n$

$\exists r_n \in (0, \frac{1}{2^n})$ $\quad \overline{B(x_n, r_n)} \subset B(x_{n-1}, r_{n-1}) \cap U_n$

Claim : $\{x_n\}$ is a Cauchy sequence

If $n, m \geq l$ $\quad x_n, x_m \in B(x_l, r_l)$ $\quad d(x_n, x_m) \leq 2r_l < \frac{1}{2^{l-1}}$

Since X is complete $\quad \exists x \in X \quad \lim_{n \to \infty} x_n = x$

For $n \in \mathbb{N}$ $\quad m \geq n$ $\quad x_m \in \overline{B(x_n, r_n)}$ (closed ball)

$\Rightarrow \quad x \in \overline{B(x_n, r_n)} \subset \bigcap_{j=1}^{n} U_n \cap W \subset U_n \cap W \quad \forall n$

$x \in \bigcap_{n=1}^{\infty} U_n \cap W \Rightarrow \bigcap_{n=1}^{\infty} U_n$ is dense.

X is NOT a countable union of nowhere dense sets.

$\bigcup_{n=1}^{\infty} E_n$ ↑ nowhere dense

goal show $\left(\bigcup_{n=1}^{\infty} E_n\right)^c \neq \emptyset$

$$\bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=1}^{\infty} \bar{E}_n$$

enough to show $\left(\bigcup_{n=1}^{\infty} \bar{E}_n\right)^c \neq \emptyset$

$$\left(\bigcup_{n=1}^{\infty} \bar{E}_n\right)^c = \bigcap_{n=1}^{\infty} \left(\bar{E}_n\right)^c \overset{i)}{\neq} \emptyset$$

$\left(\bar{E}_n\right)^c$ ← open set dense

Corollary: In a complete metric space a generic set is dense.

Pf $E \subset X$ is generic E^c = set of 1st category or meager

$$E^c = \bigcup_{n=1}^{\infty} F_n \qquad F_n \text{ nowhere dense}$$

if E is not dense $\exists x \in X$ $\exists r > 0$ s.t

$$\overline{B} = \overline{B(x,r)} \subset B(x,2r) \subset E^c$$

$$\overline{B} = \bigcup_{n=1}^{\infty} F_n \cap \overline{B}$$

$F_n \cap \overline{B}$ is nowhere dense

$$\text{int}\left(\overline{F_n \cap \overline{B}}\right) \subset \text{int}\, \overline{F_n} = \emptyset$$

$\overline{B} \subset X$ closed in a complete set $\overline{B}$ complete by previous result $*$.

The uniform boundedness principle

Let X be a Banach space, Y a normed vector space. Let $A \subset \mathcal{L}(X,Y)$. Suppose $\forall x \in X$

$$\sup_{T\in A} \|Tx\| < \infty \quad \text{then} \quad \sup_{T\in A} \|T\| < \infty$$

Pointwise bounds $\Rightarrow$ uniform bounds

Pf: $E_n = \{x \in X : \sup_{T\in A} \|Tx\| \le n\}$

$$= \bigcap_{T\in A} \{x \in X : \|Tx\| \le n\}$$

since $T \in \mathcal{L}(X,Y)$ is continuous, $\{x \in X : \|Tx\| \le n\}$ is closed.

$\bigcap_{T\in A} \{x \in X : \|Tx\| \le n\}$ is a closed set.

$E_n \subset E_{n+1}$

$X = \bigcup_{n=1}^{\infty} E_n$ by Baire category $\exists n$ s.t. E_n is not nowhere dense; i.e. $\operatorname{int} \overline{E_n} \neq \emptyset$, $\overline{E_n} = E_n$

$\exists x_0 \in E_n$ $\exists r_0 > 0$ $\overline{B(x_0, r_0)} \subset B(x_0, 2r_0) \subset E_n$

$x \in \overline{B(0, r_0)}$

$$\|Tx\| \le \|T(x + x_0)\| + \|T(-x_0)\| \le n + n = 2n$$

($x + x_0 \in \overline{B(x_0, r_0)}$)

$x \in \overline{B(0, r_0)}$ $\quad \|Tx\| \leq 2n$;

$x \in X \quad x \neq 0 \quad r_0 \dfrac{x}{\|x\|} \in \overline{B(0, r_0)}$

$$\left\| T\left(r_0 \frac{x}{\|x\|} \right) \right\| \leq 2n$$

$$\|Tx\| \leq \frac{2n}{r_0} \|x\| \qquad \forall x \in X$$

$$\Rightarrow \quad \|T\| \leq \frac{2n}{r_0} \qquad \forall T \in A.$$

Definition: Let X, Y be topological (metric) spaces $f: X \to Y$ is said to be open if $f(U)$ is open in Y whenever U is open in X.

Example: X, Y normed vector space $\|(x,y)\| = \|x\| + \|y\|$

$\pi: X \times Y \to X$

$(x,y) \mapsto \pi(x,y) = x$ is open

Definitions: ① X, Y normed vector spaces $T : X \to Y$ linear

the graph of T

$$\text{graph } T = \{(x, Tx) : x \in X\} \subset X \times Y$$

② $T : X \to Y$ linear is closed if graph T is a closed subset of $X \times Y$.