Theorem: Let X be a normed vector space

(*) i) If M is a closed subspace of X and $x \in X \setminus M$ there exists $f \in X^*$ such that $f(x) \neq 0$ & $f|_M = 0$

If $\delta = \inf_{y \in M} \|x - y\|$ f can be chosen so that $\|f\| = 1$ & $f(x) = \delta$

ii) If $x \in X$, $x \neq 0$ $\exists f \in X^*$ s.t $\|f\| = 1$ & $f(x) = \|x\|$

iii) Bounded linear functionals separate points; i.e if $x, y \in X$ $x \neq y$ $\exists f \in X^*$ such that $f(x) \neq f(y)$

(*) iv) If $x \in X$, define $\hat{x} : X^* \longrightarrow \mathbb{R}$ by $\hat{x}(f) = f(x)$.

The map $\wedge : X \longrightarrow (X^*)^* = X^{**}$ is an isometry.

(here $(X^*)^* = \mathcal{L}(X^*, \mathbb{R})$ dual of X^*)

i) If $\mathcal{M}$ is a closed subspace of X and $x \in X \setminus \mathcal{M}$

there exists $f \in X^*$ such that $f(x) \neq 0$ & $f|_{\mathcal{M}} = 0$

If $\delta = \inf_{y \in \mathcal{M}} \|x - y\|$ f can be chosen so that

$\|f\| = 1$ & $f(x) = \delta$

$y \in \mathcal{M}$

$f : \mathcal{M} + \mathbb{R}x \longrightarrow \mathbb{R}$ $\quad f(y + \lambda x) = \lambda \delta$

f is linear $\quad f(x) = \delta \quad f|_{\mathcal{M}} = 0$ in HB $p = \|\ \|$

$|f(y + \lambda x)| = |\lambda| \delta \leq |\lambda| \|\lambda^{-1} y + x\| = \|y + \lambda x\|$

$\lambda^{-1} y \in \mathcal{M}$

By HB $\quad F : X \to \mathbb{R}$

want $|F(z)| \leq \|z\|$ & $F|_{\mathcal{M} + \mathbb{R}x} = f$

$F(-z) = -F(z) \leq \|-z\| = \|z\| \Rightarrow |F(z)| \leq \|z\| \Rightarrow \|F\| \leq 1$

$\forall \varepsilon > 0 \quad \exists y_\varepsilon \in \mathcal{M} \quad \delta \leq \|y_\varepsilon - x\| \leq \delta + \varepsilon$

$F(x - y_\varepsilon) = f(x - y_\varepsilon) = \delta \geqslant \|x - y_\varepsilon\| - \varepsilon$ / $\frac{F(x - y_\varepsilon)}{\|x - y_\varepsilon\|} \geqslant 1 - \frac{\varepsilon}{\delta} \overset{\forall \varepsilon > 0}{=} 1$

$\Rightarrow \|F\| \geqslant 1$

iv) If $x \in X$, define $\hat{x} : X^* \longrightarrow \mathbb{R}$ by $\hat{x}(f) = f(x)$.

The map $\wedge : X \longrightarrow (X^*)^* = X^{**}$ is an isometry.

$$|\hat{x}(f)| = |f(x)| \leq \|f\| \|x\| \quad \Rightarrow \quad \|\hat{x}\| \leq \|x\|$$

ii) $\Rightarrow$ $x = 0$ ✓ $\qquad x \neq 0$ $\qquad \exists f \in X^*$ $\|f\| = 1$

and $\qquad f(x) = \|x\| = \hat{x}(f) \leq \|\hat{x}\| \|f\| = \|\hat{x}\|$

$$\|x\| \leq \|\hat{x}\|.$$

R problem 2

Due Monday January 25

A normed vector space $\mathcal{X}$ is **strictly convex** if whenever $x \neq y$ and $\|x\| = \|y\| = 1$ then for all $0 < t < 1$,

$$\|tx + (1-t)y\| < 1.$$

A normed vector space $\mathcal{X}$ is **uniformly convex** if for every $\epsilon > 0$ there exists $\delta \in (0,1)$ so that if

$$\|x\| = \|y\| = 1 \text{ and } \|x - y\| \geq \epsilon \text{ then } \|\frac{x+y}{2}\| < 1 - \delta.$$

Problem 0: What does it mean geometrically for a space to be *stricly convex* and *uniformly convex*?

Problem 1: Let $\mathcal{X}$ be a normed vector space. The duality map F is defined for every $x \in \mathcal{X}$ by

$$F(x) = \{f \in \mathcal{X}^* : \|f\| = \|x\| \text{ and } f(x) =< f, x >= \|x\|^2\}.$$

1. Prove that

$$F(x) = \{f \in \mathcal{X}^* : \|f\| \leq \|x\| \text{ and } f(x) =< f, x >= \|x\|^2\}.$$

   Deduce that $F(x)$ is nonempty, closed and convex.

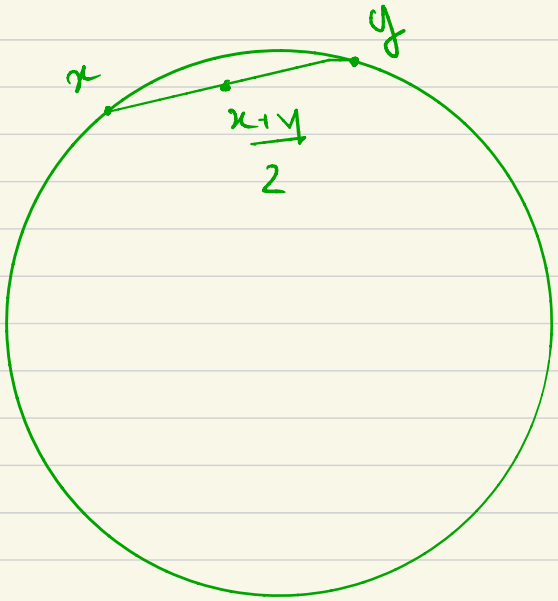
2. Prove that if $\mathcal{X}^*$ is strictly convex then $F(x)$ contains a single point. Show that in this case given $x \in \mathcal{X}$, $x \neq 0$ there exists a unique $g \in \mathcal{X}^*$ so that $\|g\| = 1$ and $g(x) = \|x\|$.

Problem 2: Let $\mathcal{X}$ be a uniformly convex normed vector space. Show that $\mathcal{X}$ is strictly convex.

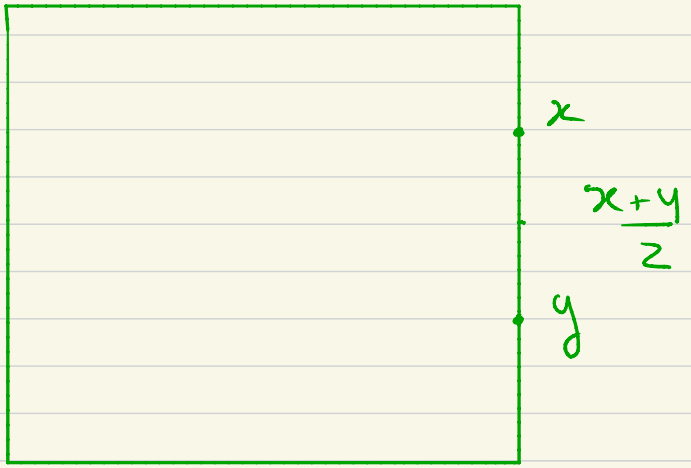
Unif convex $\uparrow$ $\forall \varepsilon > 0 \quad \exists \delta > 0$

$\|x\| = \|y\| = 1 \qquad \|x - y\| \geq \varepsilon$

$$\left\| \frac{x+y}{2} \right\| < 1 - \delta$$



unif convex



$\|x\| = \|y\| = 1$ not strictly convex

$$\left\| \frac{x+y}{2} \right\| = 1$$

strictly convex $\not\Longrightarrow$ unif convex $\qquad X = \{(x_n)_{n=1}^{\infty}\}$

$$\|x\| = \sqrt{\sum_{n=1}^{\infty} x_n^2} + \sum_{n=1}^{\infty} |x_n|$$ is strictly convex but not uniformly convex

Uniformly convex $\Rightarrow$ strictly convex

let $x, y \in X$ $\quad \|x\| = \|y\| = 1$ & $x \neq y$ by unif convex

$$\left\| \frac{x+y}{2} \right\| < 1 \qquad \text{(strict convexity } t = 1/2\text{)}$$

let $t \in (0, \frac{1}{2})$ $\quad tx + (1-t)y = 2t\left(\frac{x+y}{2}\right) + (1-2t)y$

$$\|tx + (1-t)y\| \leq 2t \left\| \frac{x+y}{2} \right\| + (1-2t)\|y\|$$

$$< 2t \cdot 1 + (1-2t) = 1$$

for $t \in (1/2, 1)$ interchange x & y and replace t by $1-t$.