Hahn - Banach Theorem

Let X be a real vector space, p a sublinear functional on X
Let M be a subspace of X and let $f : M \longrightarrow \mathbb{R}$ be a linear functional such that

$$f(x) \leq p(x) \qquad \forall x \in M$$

There exists a linear functional

$$F : X \longrightarrow \mathbb{R} \quad \text{such that}$$

i) $F(x) \leq p(x) \qquad \forall x \in X$

ii) $F|_M = f$

Step 1 : If $x_0 \in X \setminus M$ f can be extended to a linear functional $g : M + \mathbb{R}x_0$ s.t $g(x) \leq p(x)$; $g|_M = f$

Step 2 : $\mathcal{F} = \{ g : \mathcal{N} \to \mathbb{R}$, $M \subset \mathcal{N}$ ← subspace , $g|_M = f$ & $g(x) \leq p(x) \}$

$\mathcal{F} \neq \emptyset$, has a partial order, Hausdorff max principle, $\mathcal{F}$ has a maximal element

Step 1 $\quad x_0 \in X \setminus \mathcal{A} \qquad y_1, y_2 \in \mathcal{A} \qquad y_1 + y_2 \in \mathcal{A}$

$$f(y_1) + f(y_2) = f(y_1 + y_2) \leq p(y_1 + y_2)$$

$$\leq p(y_1 - x_0) + p(x_0 + y_2)$$

$$f(y_1) - p(y_1 - x_0) \leq p(x_0 + y_2) - f(y_2)$$

$$\sup_{y_1 \in \mathcal{A}} f(y_1) - p(y_1 - x_0) \leq \inf_{y_2 \in \mathcal{A}} p(x_0 + y_2) - f(y_2)$$

Choose α s.t

$$\boxed{\sup_{y \in \mathcal{A}} f(y) - p(y - x_0) \leq \alpha \leq \inf_{y \in \mathcal{A}} p(x_0 + y) - f(y)}$$

Define $\quad g : \mathcal{A} + \mathbb{R}x_0 \longrightarrow \mathbb{R} \qquad g(y + \lambda x_0) = f(y) + \lambda\alpha$

$\lambda \in \mathbb{R} \quad y \in \mathcal{A}$

g is linear

$$\sup_{y \in A} f(y) - p(y - x_0) \overset{(1)}{\leq} \alpha \overset{(2)}{\leq} \inf_{y \in A} p(x_0 + y) - f(y)$$

For $y \in A$ $\quad g(y) = f(y) \leq p(y)$

$z \in A + \mathbb{R} x_0$ $\quad z = y + \lambda x_0$ $\quad$ if $\quad$ we need to show this: $g(z) \leq p(z)$

(1) $\lambda > 0$ $\quad g(y + \lambda x_0) = f(y) + \lambda\alpha = \lambda\left(f\left(\frac{y}{\lambda}\right) + \alpha \right)$

$\leq \lambda \left(f\left(\frac{y}{\lambda}\right) + p\left(x_0 + \frac{y}{\lambda}\right) - f\left(\frac{y}{\lambda}\right) \right)$ (2) with y/λ

$\leq \lambda\, p\left(x_0 + \frac{y}{\lambda}\right) = p(\lambda x_0 + y)$

$\uparrow \lambda > 0$ $\quad$ p sublinear

(2) $\lambda < 0$ $\quad g(y + \lambda x_0) = -\lambda \left(f\left(-\frac{y}{\lambda}\right) - \alpha \right)$ $\quad$ (1) $-y/\lambda$

$\leq -\lambda\, p\left(-\frac{y}{\lambda} - x_0\right) = p(y + \lambda x_0)$

$\uparrow -\lambda > 0$ $\quad$ p subl.

$g(z) \leq p(z) \quad \forall z \in A + \mathbb{R} x_0$ $\quad$ (step 1)

Step 2 : $\mathcal{F} = \{ g : \mathcal{N} \to \mathbb{R} : \mathcal{M} \subset \mathcal{N}$ (subspace), $g|_{\mathcal{M}} = f, g \le p \} \neq \emptyset$

$\mathcal{F}$ is partially ordered $g_1 \prec g_2$ if $\mathcal{N}_1 \subset \mathcal{N}_2$ & $g_1 = g_2$ on $\mathcal{N}_1$ (g_2 is an extension of g_1)

Hausdorff maximum principle : every non-empty partially ordered set has a maximal linearly ordered family $\{g_\alpha\}$

Define $F : \bigcup_\alpha \mathcal{N}_\alpha \to \mathbb{R}$, $F(x) = g_\alpha(x)$ if $x \in \mathcal{N}_\alpha$ (*)

Facts : (1) $\bigcup_\alpha \mathcal{N}_\alpha$ is a linear space

(2) F is well defined (because by def of ordered if $x \in \mathcal{N}_\alpha \cap \mathcal{N}_\beta$ either ($\mathcal{N}_\alpha \subset \mathcal{N}_\beta$) or [$\mathcal{N}_\beta \subset \mathcal{N}_\alpha$] and ($g_\beta$ ext of g_α) or [g_α ext of g_β]

(3) $F(x) \le p(x)$ bec true for g_α (*) F ext f

(4) $F \in \mathcal{F}$, $\bigcup_\alpha \mathcal{N}_\alpha = X$ ← claim

④ $F \in \mathcal{F}$, F ext of f we want $X = \bigcup_\alpha \mathcal{N}_\alpha$

if $x_0 \in X \setminus \bigcup \mathcal{N}_\alpha$ G extension F

$$f < g_\alpha < F < G$$

$\times$ max l. o.

$X = \bigcup_\alpha \mathcal{N}_\alpha$ $F = G$, $G \in \{g_\alpha\}$

Theorem: Let X be a normed vector space

i) If M is a closed subspace of X and $x \in X \setminus M$ there exists $f \in X^*$ such that $f(x) \neq 0$ & $f|_M = 0$
If $\delta = \inf_{y \in M} \|x - y\|$ f can be chosen so that
$\|f\| = 1$ & $f(x) = \delta$

ii) If $x \in X$, $x \neq 0$ $\exists f \in X^*$ s.t $\|f\| = 1$ & $f(x) = \|x\|$ ($M = \{0\}$)

iii) Bounded linear functionals separate points; i.e if $x, y \in X$ $x \neq y$ $\exists f \in X^*$ such that $f(x) \neq f(y)$ (apply ii) to $x - y$)

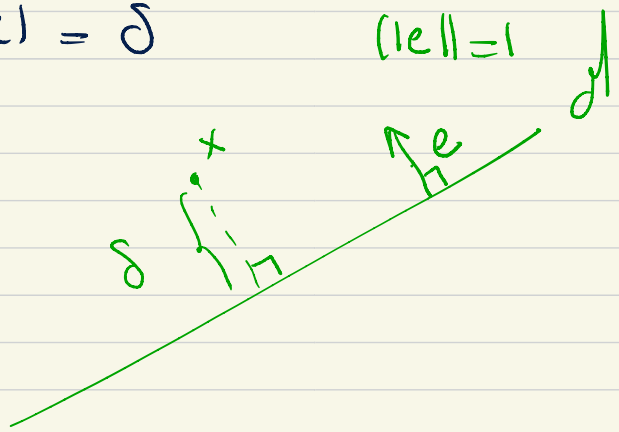
iv) If $x \in X$, define $\hat{x} : X^* \longrightarrow \mathbb{R}$ by $\hat{x}(f) = f(x)$.
The map $\wedge : X \longrightarrow (X^*)^* = X^{**}$ is an isometry.

(here $(X^*)^* = \mathcal{L}(X^*, \mathbb{R})$ dual of X^*)

(i) If $\mathcal{M}$ is a closed subspace of X and $x \in X \setminus \mathcal{M}$ there exists $f \in X^*$ such that $f(x) \neq 0$ & $f|_{\mathcal{M}} = 0$

If $\delta = \inf_{y \in \mathcal{M}} \|x - y\|$ f can be chosen so that

$\|f\| = 1$ & $f(x) = \delta$



$f(x) = \langle x, e \rangle = \delta$

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$f(y) = \langle y, e \rangle$

$|f(y)| \leq \|y\| \|e\|$

$\|f\| \leq 1$

$f(e) = \langle e, e \rangle = \|e\|^2 = 1$

(iv) If $x \in X$, define $\hat{x} : X^* \longrightarrow \mathbb{R}$ by $\hat{x}(f) = f(x)$.

The map $\wedge : X \longrightarrow (X^*)^* = X^{**}$ is an isometry.

(X: normed, X^{**}: Banach)

(here $(X^*)^* = \mathcal{L}(X^*, \mathbb{R})$ dual of X^*)

$$\|x\| = \|\hat{x}\|$$

Let $\hat{X} = \{\hat{x} : x \in X\} \subset X^{**}$; X^{**} complete

$\overline{\hat{X}}$ is a Banach space if $\hat{}$ embedds X into $\overline{\hat{X}}$ as a dense subspace. $\overline{\hat{X}}$ = completion of X

If X is Banach; then so is $\hat{X}$ (because $\hat{}$ isometry)

thus $\hat{X} \subset X^{**}$ closed and $\overline{\hat{X}} = \hat{X}$.

Definition : Let X be a Banach space X is reflexive if $X^{**} = X$.