Definitions: (1) A linear map $T: X \to Y$ between 2 normed vector spaces is bounded if there exists $C \geq 0$ such that

$$\|Tx\|_Y \leq C\|x\|_X \qquad \forall x \in X$$

(2) A linear map $T: X \to Y$ between 2 normed vector spaces is continuous if given $\varepsilon > 0$ there is $\delta > 0$ such that

$$\|Tf - Tg\|_Y < \varepsilon \quad \text{whenever} \quad \|f - g\|_X < \delta$$

Proposition: Let X, Y be normed vector spaces & $T: X \to Y$ linear map. T.f.a.e

1) T is continuous
2) T is continuous at 0
3) T is bounded

Notation: If X, Y normed vector spaces

$\mathcal{L}(X, Y) = \{T: X \to Y$ bounded linear map$\}$ vector space.

1) $\Longrightarrow$ 2) $\Longrightarrow$ 3) $\Longrightarrow$ 1)

T cont at 0 $\qquad$ T is bounded

$\varepsilon = 1 \quad \exists \delta > 0$ s.t $\underbrace{\|f - 0\|_X}_{} < \delta$ then $\|Tf - \underset{0}{T0}\|_Y \leq 1$

if $f \in B^X(0,\delta) \Longrightarrow Tf \in B^Y(0,1)$

if $z \in X \quad z \neq 0 \quad f = \frac{z}{\|z\|} \cdot \frac{\delta}{2} \in B^X(0,\delta)$

$\|Tf\|_Y \leq 1$

$$Tf = \frac{\delta}{2} \frac{1}{\|z\|} Tz$$

$\Longrightarrow \|Tz\| \leq \frac{2}{\delta} \|z\|$

3) $\Longrightarrow$ 1) $\quad \|Tx\| \leq C\|x\| \quad \forall x \in X$

$$\|Tx - Ty\| = \|T(x-y)\| \leq C\|x-y\|$$

For $T \in \mathcal{L}(X,Y)$ let

$$\|T\| = \sup \{ \|Tx\| : \|x\| = 1 \}$$

$$= \sup \left\{ \frac{\|Tx\|}{\|x\|} : x \neq 0 \right\}$$

$$= \inf \{ C : \|Tx\| \leq C\|x\| \quad \forall x \in X \}$$

$\|Tx\| \leq \|T\| \|x\|_X$

$\|T\|$ defines a norm in $\mathcal{L}(X,Y)$.

Proposition: If Y is a Banach space so is $\mathcal{L}(X,Y)$.

Pf $\{T_n\} \subset \mathcal{L}(X,Y)$ Cauchy seq, $x \in X$ $\{T_n x\} \subset Y$ is Cauchy

$$\|T_n x - T_m x\|_Y = \|(T_n - T_m)(x)\|_Y \leq \|T_n - T_m\| \|x\|_X$$

$\|T_n x - Tx\|_Y \xrightarrow[n \to \infty]{} 0$

Since Y is complete $\{T_n x\}$ converges to $Tx = \lim_{n \to \infty} T_n x$

Claim 1: $T \in \mathcal{L}(X,Y)$ Claim 2: $\|T_n - T\| \xrightarrow[n \to \infty]{} 0$

Claim 1 : T linear ✓

T bounded : $\{T_n\}$ Cauchy $\Rightarrow$ bounded

$$\sup_n \|T_n\| \le M < \infty$$

$$\|Tx\| \le \|(T - T_n)x\| + \|T_n x\| \le \|Tx - T_n x\|_y + M\|x\|$$

$n \to \infty$ $\qquad \|Tx\| \le M\|x\|$ ← $\qquad (T_n x \to Tx)$

Claim 2 : Goal $\|T - T_n\| \to 0 \quad n \to \infty$

$$\|T - T_n\| = \sup\{\|(T - T_n)(x)\| : \|x\| = 1\} \quad \text{for each } m \in \mathbb{N}$$

$x_m^n \in X \quad \|x_m^n\| = 1 \quad$ & $\quad \|T - T_n\| \le |Tx_m^n - T_n x_m^n| + \frac{1}{2^m}$

$$\lim_{j \to \infty} T_j x_m^n = Tx_m^n$$

$$\|T - T_n\| \le |Tx_m^n - T_j x_m^n| + |T_j x_m^n - T_n x_m^n| + \frac{1}{2^m}$$

$$\|T_n - T\| \le |Tx_m^n - T_j x_m^n| + \|T_j - T_n\| \underbrace{\|x_m^n\|}_{1} + \frac{1}{2^m}$$

$$\|T_n - T\| \leq |Tx_m^n - T_j x_m^n| + \|T_j - T_n\| \underbrace{\|x_m^n\|}_{1} + \frac{1}{2^m}$$

for $j \geq n_0$ $\quad |T_j x_m^n - T x_m^n| < 1/2m$

$$\|T_n - T\| \leq \underbrace{\|T_j - T_n\|}_{< 1/2m} + \frac{2}{2^m}$$

$\exists n_1$ s.t $j, n \geq n_1$ $\qquad \forall m \ \exists n_1$ s.t $n \geq n_1$

$$\|T_n - T\| \leq 3/2m$$

$$\lim_{n \to \infty} \|T - T_n\| = 0$$

normed vector

Corollary: Let $\mathcal{B}$ be a (Banach) space the dual of $\mathcal{B}$,

$$\mathcal{B}^* = \mathcal{L}(\mathcal{B}, \mathbb{R}) \text{ is a Banach space}$$

Remarks: ① $T \in \mathcal{L}(X, Y)$, $S \in \mathcal{L}(Y, Z)$ then $ST \in \mathcal{L}(X, Z)$

② $T \in \mathcal{L}(X, Y)$ is invertible (or isomorphism) if T is bijective (*) and T^{-1} is a bounded linear map; ie

$\exists\, C_0 > 0$ s.t $\|Tx\| \geq C_0 \|x\| \quad \forall x \in X$

T is an isometry if $\|Tx\| = \|x\|$

Definitions: ① Let X be a vector space. A linear map from X to $\mathbb{R}$ (or $\mathbb{C}$) is called a linear functional.

② $\mathcal{L}(X, \mathbb{R})$ is the dual space of X, it is denoted by X^* = Banach space with operator norm.

Examples ?

Examples: $\begin{cases} X = \mathbb{R}^n \\ X^* = X \end{cases}$ $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$ ✓

$X = L^1(\mathbb{R}^n)$ $X^* \overset{?}{=} L^\infty(\mathbb{R}^n)$

Hahn - Banach Theorem

Existence of non zero bounded linear functionals !

Definition: Let X be a real vector space, a sublinear functional on X is a map $p: X \to \mathbb{R}$ such that

i) $p(x+y) \leq p(x) + p(y) \qquad \forall x, y \in X$

ii) $p(\lambda x) = \lambda\, p(x) \qquad \forall x \in X \qquad \forall \lambda \geq 0$

Example: $(X, \| \ \|)$ $\| \ \|$ norm or semi-norm

Hahn - Banach Theorem

Let X be a real vector space, p a sublinear functional on X
Let M be a subspace of X and let $f : M \longrightarrow \mathbb{R}$ be a linear functional such that

$$f(x) \leq p(x) \qquad \forall x \in M$$

There exists a linear functional

$$F : X \longrightarrow \mathbb{R} \quad \text{such that}$$

i) $F(x) \leq p(x) \qquad \forall x \in X$

ii) $F|_M = f$

Step 1 : If $x_0 \in X \setminus M$ f can be extended to a linear functional $g : M + \mathbb{R}x_0$ s.t $g(x) \leq p(x)$; $g|_M = f$

Step 2 : $\mathcal{F} = \{ g : \mathcal{N} \to \mathbb{R}$, $M \subset \mathcal{N}$ ← subspace , $g|_M = f$ & $g(x) \leq p(x) \}$

$\mathcal{F} \neq \emptyset$, has a partial order, Hausdorff max principle, $\mathcal{F}$ has a maximal element