Elements of functional analysis

Definitions (a) A normed vector space X (over $\mathbb{C}$ or $\mathbb{R}$) is a vector space with a norm $\|\cdot\| : X \longrightarrow \mathbb{R}$ s.t.

(1) $\|x\| \geq 0 \quad \forall x \in X \quad$ & $\|x\| = 0$ iff $x = 0$

(2) $\|\lambda x\| = |\lambda| \, \|x\| \quad \forall x \in X \quad \forall \lambda \in \mathbb{R}$ (or $\mathbb{C}$)

(3) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

(b) If $\|\ \|$ only satisfies (2) & (3) it is called a semi-norm

(c) Two norms $\|\ \|_1$ & $\|\ \|_2$ on X are equivalent if

$\exists C > 1$ s.t.

$$C^{-1} \|x\|_1 \leq \|x\|_2 \leq C \|x\|_1 \qquad \forall x \in X$$

(d) A complete normed vector space is called a BANACH space

Examples (c) $(\mathbb{R}^n, \|\ \|_\infty)$ $\quad \|x\|_\infty = \sup_{1 \leq i \leq n} \{|x_i| : x = (x_1 \dots x_n)\}$

$(\mathbb{R}^n, \|\ \|_2)$ $\quad \|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$

all norms in $\mathbb{R}^n$ are equivalent (check)

$$\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\, \|x\|_\infty$$

(d) $L^1(\mu)$ is a Banach space.

(e) Some other spaces of functions

$$L^p(\mu) = \left\{ f : X \to \mathbb{R} : \int |f|^p \, d\mu < \infty \right\}$$

$$\|f\|_p = \left(\int |f|^p \, d\mu \right)^{1/p}$$

Definitions: ① Let $(X, \|\ \|)$ be a normed vector space, a series $\sum_{n=1}^{\infty} x_n$ is said to converge to x if

$$\left\| \sum_{n=1}^{N} x_n - x \right\| \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty$$

(notation : $x = \sum_{n=1}^{\infty} x_n$)

② $\sum_{n=1}^{\infty} x_n$ is said to converge absolutely if

$$\sum_{n=1}^{\infty} \|x_n\| < \infty$$

Theorem: A normed vector space $(X, \|\ \|)$ is complete iff every absolutely convergent series in X, converges to an element in X.

Examples: We use this to show L^1 is complete

BUILDING NORMED VECTOR SPACES FROM NORMED VECTOR SPACES

① Product spaces

Let $(X, \|\ \|_X)$ & $(Y, \|\ \|_Y)$ be normed vector spaces then $(X \times Y, \|\ \|)$ is a normed vector space with

$$\|(x,y)\|_1 = \max\{\|x\|_X, \|y\|_Y\}$$

or

$$\|(x,y)\|_2 = \|x\|_X + \|y\|_Y$$

or

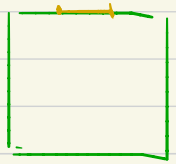
$$\|(x,y)\|_3 = \sqrt{\|x\|_X^2 + \|y\|_Y^2}$$

Remark: all of this norms are equivalent.

$(\mathbb{R}^2, \|\ \|_\infty)$

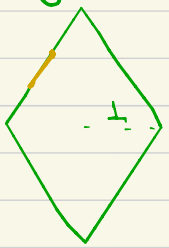
unit ball

$B = \{x : \|x\| \leq 1\}$

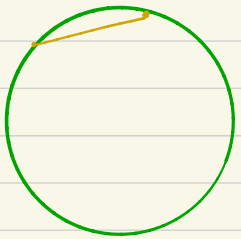


$(\mathbb{R}^2, \|\ \|_1)$

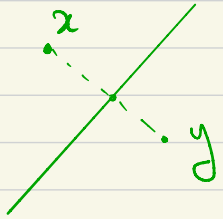
$\|(x,y)\| = \|x\| + \|y\|$



$(\mathbb{R}^2, \|\ \|_2)$



(2) Quotient spaces



Let X be a vector space, $\mathcal{M} \subset X$ a vector subspace

For $x, y \in X$ define $x \sim y$ iff $x - y \in \mathcal{M}$

↑ equivalence relation

$x + \mathcal{M}$ equiv class of x in $X/\mathcal{M}$

$$(x+\mathcal{M}) + (y+\mathcal{M}) = x+y+\mathcal{M}$$

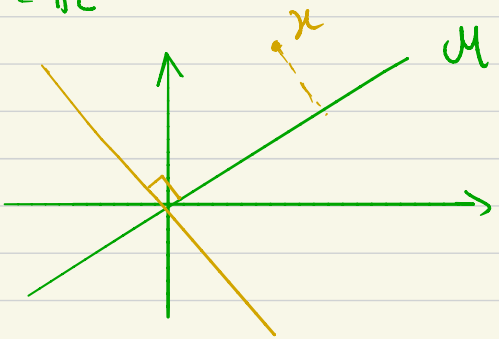
$$\lambda(x+\mathcal{M}) = \lambda x + \mathcal{M}$$

If $(X, \|\ \|)$ &

if $\mathcal{M}$ is closed then $X/\mathcal{M}$ inherits a norm from X the quotient norm

$$\|x+\mathcal{M}\| = \inf_{y \in \mathcal{M}} \|x+y\| = \text{"distance from } x \text{ to } \mathcal{M}\text{"}.$$

$X = \mathbb{R}^2$



③ Spaces of functions on metric spaces

Let (X, d) be a metric space, define

$$\mathcal{B}(X) = \{ f : X \to \mathbb{R} \text{ bounded} \}$$

$$\mathcal{C}(X) = \{ f : X \to \mathbb{R} : \text{continuous} \}$$

$$\mathcal{BC}(X) = \mathcal{B}(X) \cap \mathcal{C}(X)$$

If $f \in \mathcal{B}(X)$ define $\|f\|_u = \sup \{ |f(x)| : x \in X \}$

↑ uniform norm

$$\rho(f, g) = \|f - g\|_u \quad \text{norm on } \mathcal{B}(X)$$

Proposition: $(\mathcal{B}(X), \| \|_u)$ is a complete metric space. Moreover $\mathcal{BC}(X)$ is a closed subspace of $\mathcal{B}(X)$ in the uniform norm. In particular $\mathcal{BC}(X)$ is complete

(i.e $(\mathcal{B}(X), \| \|_u)$ is a complete normed vector space)

① $\mathcal{F} \subset \mathcal{C}(X)$ is **equicontinuous at** $x \in X$ if

$\forall \varepsilon > 0 \quad \exists \delta > 0$. s.t $|f(x) - f(y)| < \varepsilon$ whenever $d(x,y) < \delta$

for all $f \in \mathcal{F}$

② $\mathcal{F} \subset \mathcal{C}(X)$ is **equicontinuous** if it is equicontinuous at each $x \in X$.

③ $\mathcal{F} \subset \mathcal{C}(X)$ is **pointwise bounded** if

$\{ f(x) : f \in \mathcal{F} \}$ is a bounded subset of $\mathbb{R}$

Theorem: (Arzela - Ascoli) Let (X, d) compact separable metric space. If $\mathcal{F} \subset \mathcal{C}(X)$ is equicontinuous and pointwise bounded then given a sequence $\{f_n\} \subset \mathcal{F}$ there exists a subsequence $\{f_{n_k}\}$ and a function $f \in \mathcal{C}(X)$ such that $f_{n_k} \to f$ in the uniform norm.

Corollary: If $\mathcal{F} \subset \mathcal{C}(\mathbb{R}^d)$ is equicontinuous and pointwise bounded and $\{f_n\} \subset \mathcal{F}$ there exists a subsequence $\{f_{n_k}\}$ and $f \in \mathcal{C}(\mathbb{R}^d)$ such that $f_{n_k}(x) \to f(x)$ $\forall x \in \mathbb{R}^d$ and $f_{n_k} \to f$ uniformly on compact sets. $n_k \to \infty$

Problem 2: Let (X, d) be a compact separable metric space. Let $\mathcal{D} \subset X$ be a countable dense set. Let $\mathcal{F} \subset \mathcal{C}(X)$ be equicontinuous and pointwise bounded. Let $\{f_n\}_n \subset \mathcal{F}$.

1. Prove that there exists a subsequence $\{f_{n_k}\}_k \subset \{f_n\}_n$ which converges for every $x \in \mathcal{D}$.
2. Prove that $\{f_{n_k}\}_k$ converges pointwise in X to a continuous function f.
3. Show that $\{f_{n_k}\}_k$ converges uniformly in X to f above.