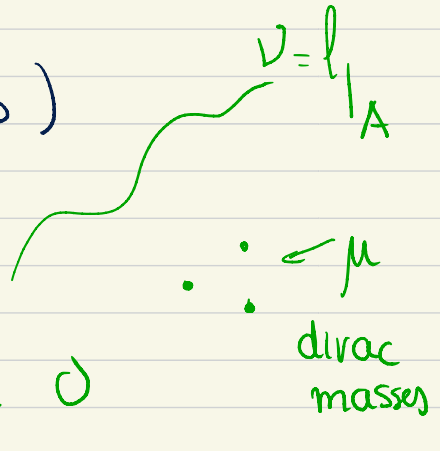
Definitions: ① Two signed measures μ & ν on $(X, \mathcal{M})$ are mutually singular if there are disjoint sets A & B in $\mathcal{M}$ ($A \cap B = \emptyset$) so that $\forall E \in \mathcal{M}$

$$\nu(E) = \nu(E \cap A) \quad \& \quad \mu E = \mu(E \cap B)$$



Notation $\mu \perp \nu$

Remark $\nu(B) = \nu(B \cap A) = 0$ & $\mu A = 0$

② For a signed measure ν and a non-negative measure μ on $(X, \mathcal{M})$ we say that ν is absolutely continuous w.r.t μ if for $E \in \mathcal{M}$

$$\mu E = 0 \implies \nu(E) = 0$$

Notation $\nu \ll \mu$

Examples: μ measure $f \in L^1(\mu)$ ($\int f^- d\mu < \infty$ or $\int f^+ d\mu < \infty$ enough)

$\nu \ll \mu$ $\quad \nu(E) = \int_E f \, d\mu$

↳ ν is finite, $|\nu|$ is also finite

(X, d) metric space $\mathcal{M} = \mathcal{B}$ μ measure on $(X, \mathcal{M})$

then $f \in L^1_{loc}(X)$ if $\forall B$ ball

$$\int_B |f| \, d\mu < \infty$$

Absolute continuity of measure before today

$(X, \mathcal{M}, \mu)$ measure space $f \in L^1(\mu)$

$\forall \varepsilon > 0$ $\exists \delta > 0$ s.t $\mu E < \delta$ then $\int_E |f| \, d\mu < \varepsilon$

Q: What did we call absolute continuity of measure before?

Proposition: Let ν be a signed measure and μ be a measure on $(X, \mathcal{A})$ then

i) if $\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t if $\mu E < \delta$ then $|\nu(E)| < \varepsilon$ then $\nu \ll \mu$ (if $\mu E = 0 < \delta \ \forall \delta > 0$, $\forall \varepsilon > 0$ this satisfied $|\nu(E)| < \varepsilon \ \forall \varepsilon > 0$) $\Downarrow \nu(E) = 0$

ii) if $|\nu|$ is finite then $\nu \ll \mu$ implies that $\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t $\mu E < \delta \Rightarrow |\nu(E)| < \varepsilon$

iii) $\nu \ll \mu$ iff $|\nu| \ll \mu$

ii) if $|\nu|$ is finite then $\nu \ll \mu$ implies that

$\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t $\mu E < \delta \Rightarrow |\nu(E)| < \varepsilon$

iii) $\nu \ll \mu$ iff $|\nu| \ll \mu$

iii) $\Leftarrow$ assume $\mu E = 0 \Rightarrow |\nu|(E) = \nu^+(E) + \nu^-(E) = 0$

$\Rightarrow \nu^+(E) = \nu^-(E) = 0 \Rightarrow \nu(E) = \nu^+(E) - \nu^-(E) = 0$

$\Rightarrow$ $E \in \mathcal{A}$ $\mu E = 0$ $E = \bigcup_j E_j$ $E_j \cap E_i = 0$

$\mu \geq 0$ $\mu E_j = 0$

$\Rightarrow \nu(E_j) = 0$

$$|\nu|(E) = \sup \left\{ \sum_j |\nu(E_j)| : \quad E = \bigcup E_j \quad E_i \cap E_j = \emptyset \right\}$$

$= 0$

(ii) $|\nu| = \nu^+ + \nu^-$ finite; $\nu^\pm$ finite since $\nu \ll \mu$

then $|\nu| \ll \mu$ $(|\nu(E)| \leq |\nu|(E))$

ii) if ~~$|\nu|$ is finite~~ ν measure finite then $\nu \ll \mu$ implies that

$\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t $\mu E < \delta \Rightarrow \nu(E) < \varepsilon \qquad (*)$

ν finite measure, $\nu \ll \mu \Rightarrow (*)$

$\neg(*)$ = $(*)$ does not hold : $\exists \varepsilon_0 > 0$ s.t $\forall n \in \mathbb{N} \quad \delta_n = \frac{1}{2^n}$

$\exists E_n \in \mathcal{A} \quad \mu E_n < 1/2^n$ & $\nu(E_n) \geq \varepsilon_0$

$F_k = \bigcup_{n=k}^{\infty} E_n \qquad F_k \subset F_{k-1} \subset \dots \subset F_1 \qquad \nu(F_1) < \infty$

$\mu F_k \leq \sum_{n \geq k} \mu E_n < \sum_{n \geq k} \frac{1}{2^n} = \frac{1}{2^{k-1}} \qquad \mu F_1 \leq 2$

$\mu(\cap F_k) = \lim \mu F_k = 0$

$\nu(F_k) \geq \nu(E_k) \geq \varepsilon_0$

$\mu(F_k) \to 0$ & $\mu(\cap_k F_k) = 0$

$\nu(\cap F_k) = \lim \nu(E_k) \geq \varepsilon_0$ $\Rightarrow$ $*$ contradiction

Theorem (Radon - Nikodym)

Suppose μ is σ-finite (non-negative) measure on $(X, \mathcal{M})$ and ν is a σ-finite signed measure then there exist unique signed measures ν_a & ν_s on $(X, \mathcal{M})$ s.t

$$\nu_a \ll \mu \quad \& \quad \nu_s \perp \mu$$

ν_a: absolutely continuous part; ν_s: singular part

$$\nu = \nu_a + \nu_s$$

Moreover there exists a μ-measurable function f s.t either

$$\int f^+ \, d\mu < \infty \quad \text{or} \quad \int f^- \, d\mu < \infty \quad \text{and}$$

$$\nu_a(E) = \int_E f \, d\mu \qquad \forall \, E \in \mathcal{M}$$

Remarks: ① if $\nu \ll \mu$ then $d\nu = f d\mu$ (fundamental theorem of calculus for measures); if moreover ν is a measure then $f \geq 0$ μ-a.e

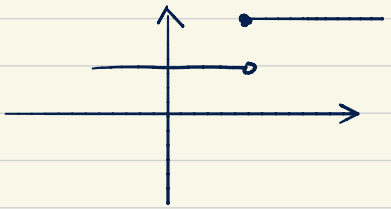
② $F : [0,1] \to \mathbb{R}$ if F increasing (hw)

$$F = F_A + F_S + F_J$$

F_A: absolutely continuous

F_S: singular, $F_S' = 0$ a.e

F_J: "jump function"



F gives rise to a measure

$$F' = F_A' + \cancel{F_S'} + F_J' \text{ "}$$

($F_S' = 0$)

$$[DF] = F_A' \, dm + [DF]_S$$

$[DF]$: measure

$F_A' \, dm$: $[DF]_a$

$[DF]_S \perp m$

$[DF]_S$: "Drac measures at the jumps"

$$[DF](B) = \int_B F_A' \, dm + [DF]_S(B)$$

† Cantor function $[0,1]$

$f' = 0$ a.e

$f, g \in C^1[0,1]$ $\qquad$ $g \in C_c^1(0,1)$

$$\int_0^1 f g' + g f' = \int_0^1 (fg)' \, dx = fg(1) - fg(0) = 0$$

$$\int_0^1 f g' \, dx = -\int_0^1 g \, f'(x) dx$$

If $f \in BV[0,1]$ $\qquad$ $\forall g \in C_c^1(0,1)$

$$\int_0^1 f(x) g'(x) dx = -\int_{[0,1]} g \, d[Df]$$

Corollary: Let ν be a measure on $\mathbb{R}^n$ s.t $\nu(K) < \infty$ for every compact set $K \subset\subset \mathbb{R}^n$. Assume that $\nu \ll m$ (m Lebesgue measure) then

$$d\nu = f\,dm \quad ; \quad f \in L^1_{loc}(\mathbb{R}^n) \quad \text{and}$$

$$f(x) = \lim_{r \to 0} \frac{\nu(B(x,r))}{m(B(x,r))} \qquad \text{a.e } x \in \mathbb{R}^n$$

Pf: $\nu \ll m$ $\quad d\nu = f\,dm$ $\quad f \geq 0$ $\quad m$ a.e

$$\nu(B(x,r)) = \int_{B(x,r)} f\,dm < \infty \qquad f \in L^1_{loc}(\mathbb{R}^n)$$

By Lebesgue differentiation

$$\frac{\nu(B(x,r))}{m(B(x,r))} \xrightarrow[r \to 0]{} f(x) \qquad m \text{ a.e } x \in \mathbb{R}^n.$$