# Differentiation of measures

Q: Given 2 measures on the same measurable space, how do they relate to each other?

## Absolute continuity of measures

Let $(X, \mathcal{M})$ be a measurable space

**Definition**: A signed measure $\nu$ on a $\sigma$-algebra $\mathcal{M}$ over a set $X$ is a mapping $\nu : \mathcal{M} \to [-\infty, \infty]$ that satisfies

i) $\nu(\phi) = 0$

ii) $\nu$ assumes at most one value $\pm\infty$

iii) If $\{E_j\}_j \subset \mathcal{M}$ disjoint family then

$$\nu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \nu(E_j)$$

$\uparrow_{\infty}$

for this equality to hold $\sum_{j=1}^{\infty} \nu(E_j)$ must be independent of the rearrangements of terms. If $|\nu(\bigcup E_j)| < \infty$ then the sum converges absolutely.

If $\sum_{i=1}^{\infty} a_i$ converges but $\sum_{i=1}^{\infty} |a_i| = +\infty$ then

$\{a_i\}$

converges conditionally

If $\sum_{i=1}^{\infty} a_i$ conv. conditionally, you can rearrange $\{a_i\}$

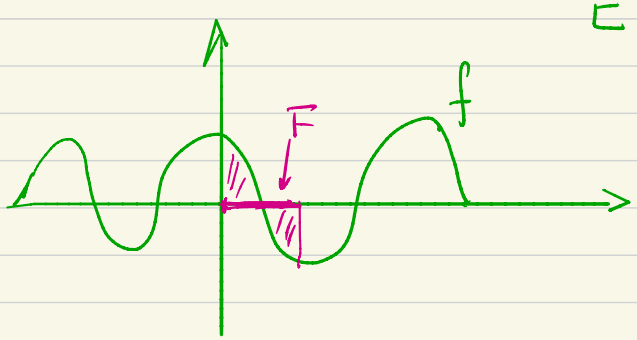
$\{a_{\sigma(i)}\}_i$ if $\sum_{i=1}^{\infty} a_{\sigma(i)}$ converges to any prescribed

value.

Examples: $(X, \mathcal{M}, \mu)$ measure ($\mu$ is measure, $\mu \geq 0$)

$f$ measurable : $f: X \to \overline{\mathbb{R}}$ & either $\int f^+ d\mu < \infty$ or $\int f^- d\mu < \infty$

then $\nu(E) = \int_E f \, d\mu$ is a signed measure




$(\mathbb{R}, \mathcal{L}, m)$ $\quad \nu(E) = \int_E f \, dm$

$E \subset \mathbb{R}$

If $\nu$ is a signed measure what does it mean that set $E$ has $\nu$ measure 0 ?

$$\nu(F) = \int_F f \, dm = 0$$

**Definition**: Let $\nu$ be a signed measure on $(X, \mathcal{M})$. We define the total variation of $\nu$ by

$$|\nu|(E) = \sup \left\{ \sum_{j=1}^{\infty} |\nu(E_j)| : E_j \in \mathcal{M} \quad E = \bigcup_{j=1}^{\infty} E_j \right.$$

$$\left. \text{if } E_j \cap E_i = \emptyset \text{ if } i \neq j \right\}$$

If $\nu \equiv +\infty$ then $|\nu| = +\infty$, we assume $\nu \not\equiv +\infty$

**Proposition**: The total variation $|\nu|$ of a signed measure $\nu$ is itself a measure that satisfies

$$|\nu(E)| \leq |\nu|(E) \qquad \forall E \in \mathcal{M}.$$

Assume $\nu$ finite then define

$$\nu^+ = \frac{1}{2}(|\nu| + \nu) \quad \text{positive variation}$$

Q: $\nu(E) = \int_E f\,d\mu$ is $|\nu|(E) \overset{?}{=} \int_E |f|\,d\mu$

$\mu$ measure

$$\nu^- = \frac{1}{2}(|\nu| - \nu) \quad \text{negative variation}$$

Remark: $\nu^\pm$ are measures, they satisfy

$$\nu = \nu^+ - \nu^- \quad \& \quad |\nu| = \nu^+ + \nu^-$$

If $\nu$ $\sigma$-finite $X = \bigcup_{j=1}^{\infty} E_j$ $E_i \cap E_j = \emptyset$ & $|\nu|(E_j) < \infty$

$$\nu^+ = \sum_j (\nu \llcorner E_j)^+ \quad \& \quad \nu^- = \sum (\nu \llcorner E_j)^-$$

Definition: A signed measure $\nu$ is $\sigma$-finite if $|\nu|$ is $\sigma$-finite. In this case so are $\nu^+$ & $\nu^-$

(if $\nu$ is finite so is $|\nu|$)

$$\nu(E) = \int_E f \, d\mu$$

$$E = E_+ \cup E_-$$

$$E_+ = \{x \in E : f(x) \geq 0\}$$

$$E_- = \{x \in E : f(x) < 0\}$$

$$|\nu|(E) \geq |\nu(E_+)| + |\nu(E_-)| = \int_E f^+ \, d\mu + \int_E f^- \, d\mu$$

$$|\nu|(E) \leq \int_E |f| \, d\mu$$

$$|\nu|(E) = \int_E |f| \, d\mu$$

**Definition** : Let $\mu$ be a signed measure on $(X, \mathcal{M})$, we say that $\mu$ is supported on $A \in \mathcal{M}$ if $\forall E \in \mathcal{M}$

$$\mu E = \mu(E \cap A)$$

**Remark & example**

① $A = \mathbb{R} \setminus \mathbb{Q}$ $\qquad \mu = \mathcal{L}^1 \llcorner A$ (restricted) ie

$$\mu(E) = \mathcal{L}^1(E \cap A) = L^1(E)$$

② $\nu(E) = \int_E f \, d\mu \qquad f \in L^1(m)$ $f$ with compact support

then $\nu$ is supported on the support of $f$.

③ graph $f = \Gamma = \{(x, f(x)) : x \in \mathbb{R}\}$




$$\mu(E) = l(E \cap \Gamma) \quad \text{length}$$