Absolutely continuous functions

Definition: A function $F : [a,b] \to \mathbb{R}$ is absolutely continuous if $\forall \varepsilon > 0$ $\exists \delta > 0$ so that

$$\sum_{k=1}^{N} |F(b_k) - F(a_k)| < \varepsilon \quad \text{whenever} \quad \sum_{k=1}^{N} (b_k - a_k) < \delta$$

and $\{(a_k, b_k)\}_{k=1}^{N}$ are disjoint intervals in $[a,b]$.

Remarks:

① If F is absolutely continuous then F is continuous.

② If F is absolutely continuous in $[a,b]$ then $F \in BV[a,b]$.

③ If f is absolutely continuous on $[a,b]$ then so is $T_a^x(f)$.

④ If $f \in L^1[a,b]$ & $F(x) = \int_a^x f(t)\,dt$ then F is absolutely continuous.

Recall that if $f \in L^1[a,b]$ $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t

$$\int_E |f(s)|\,ds < \varepsilon \quad \text{whenever} \quad mE < \delta$$

$\{(a_i,b_i)\}_{i=1}^m$ s.t $E = \bigcup_{i=1}^m (a_i,b_i)$ disjoint

s.t $\sum_{i=1}^m (b_i - a_i) < \delta \iff mE < \delta$

$$\sum_{i=1}^m |F(b_i) - F(a_i)| = \sum_{i=1}^m \left| \int_a^{b_i} f(s)\,ds - \int_a^{a_i} f(s)\,ds \right|$$

$$= \sum_{i=1}^m \left| \int_{a_i}^{b_i} f(s)\,ds \right| \leq \sum_{i=1}^m \int_{a_i}^{b_i} |f(s)|\,ds$$

$$\leq \int_{\bigcup_{i=1}^m (a_i,b_i)} |f(s)|\,ds = \int_E |f(s)|\,ds < \varepsilon$$

Theorem: If F is absolutely continuous on $[a,b]$, then F' exists a.e. $x \in [a,b]$. Moreover if $F'(x) = 0$ a.e $x \in [a,b]$ then F is constant on $[a,b]$.

Pf $F \in AC[a,b] \Rightarrow F \in BV[a,b] \Longrightarrow F'$ exists a.e $x \in [a,b]$

we need to show if $F' = 0$ a.e

$$F(x) = F(a) \quad \forall x \in [a,b]$$

enough to show $F(b) = F(a)$ when $F' = 0$ a.e.

$F \in AC$

$$E = \{ x \in (a,b) : F'(x) \text{ exists } \& F'(x) = 0 \}$$

$$mE = b - a \qquad \forall x \in E$$

$$\lim_{h \to 0} \frac{|F(x+h) - F(x)|}{|h|} = 0$$

$\forall \varepsilon > 0 \quad \exists h_{\varepsilon,x} > 0$ s.t

$h \in (-h_{\varepsilon,x}, h_{\varepsilon,x})$

$$|F(x+h) - F(x)| < \frac{\varepsilon}{4} |h| \quad \text{if } (x - h_{\varepsilon,x}, x + h_{\varepsilon,x}) \subset (a,b)$$

$\mathcal{B} = \{ [x-h, x+h] \}_{x \in E \,;\, 0 < h < \frac{h_{\varepsilon,x}}{2}}$ fine cover of E

$[x-h, x+h] \subset (a,b)$

$(mE < \infty)$

$y, z \in [x-h, x+h]$

① $|F(y) - F(z)| \leq |F(y) - F(x)| + |F(x) - F(z)|$

$\leq \frac{\varepsilon}{4}|y-x| + \frac{\varepsilon}{4}|x-z| \leq \frac{\varepsilon}{2}(2h)$

Given $\delta > 0$ there is a disjoint finite collection

$I_j = [x_j - h_j, x_j + h_j] \in \mathcal{B}$ s.t

$(\star)$ $m(E \setminus \bigcup_{j=1}^{m} I_j) < \delta$

$m([a,b] \setminus \bigcup_{j=1}^{m} I_j) = m(E \setminus \bigcup_{j=1}^{m} I_j) < \delta$

$[a,b] \setminus \bigcup_{j=1}^{m} I_j = \bigcup_{i=1}^{n} (a_i, b_i)$ disjoint

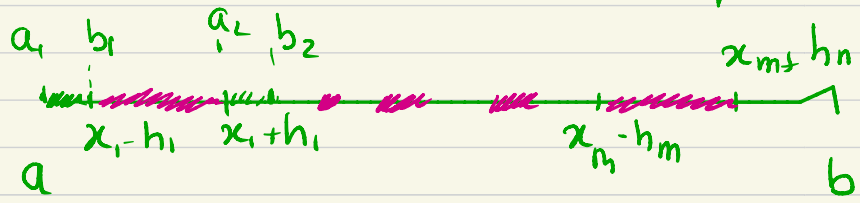
$\sum_{i=1}^{n} (b_i - a_i) < \delta$ (choose $\delta > 0$ s.t by AC

② $\sum_{i=1}^{n} |F(b_i) - F(a_i)| < \varepsilon/4$

$$|F(b) - F(a)| \le \sum_{i=1}^{n} |F(b_i) - F(a_i)| + \sum_{j=1}^{m} |F(x_j + h_j) - F(x_j - h_j)|$$

$$\sum_{i=1}^{n} |F(b_i) - F(a_i)| < \frac{\varepsilon}{4}$$

$$|F(x_j + h_j) - F(x_j - h_j)| < \frac{\varepsilon}{4}(2h_j)$$



$$|F(b) - F(a)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \underbrace{\sum_{j=1}^{m} 2h_j}_{(b-a)} \le \frac{\varepsilon}{4}(1 + (b-a)) \qquad \forall \varepsilon > 0$$

$\varepsilon \to 0$ $\qquad F(b) = F(a)$

Theorem: Suppose F is absolutely continuous on $[a,b]$. Then F' exists a.e on $[a,b]$ and $F' \in L^1[a,b]$. Moreover $\forall x \in [a,b]$

$$F(x) - F(a) = \int_a^x F'(s)\, ds$$

In particular $$\int_a^b F'(s)\, ds = F(b) - F(a).$$

Conversely if $f \in L^1[a,b]$ there exists F absolutely continuous on $[a,b]$ such that $F'(x) = f(x)$ a.e $x \in [a,b]$

$$F(x) = \int_a^x f(s)\, ds$$

Pf: $F \in AC[a,b] \Rightarrow F \in BV[a,b]$

$F = F_1 - F_2$ $\quad F_i$ bounded $\quad F_i \nearrow$ $\quad F_i' \geq 0$ a.e

$\rightarrow F_1(x) = P_a^x(f) + F(a) \qquad F_2 = N_a^x(f)$

$$0 \leq \int_a^b F_i'(s)\,ds \leq F_i(b) - F_i(a)$$

$$\int_a^b |F'(s)|\,ds \leq \int_a^b F_1'(s)\,ds + \int_a^b F_2'(s)\,ds$$

$$\leq F_1(b) - F_1(a) + F_2(b) - F_2(a)$$

$$\leq P_a^b(f) + N_a^b(f) = T_a^b(f) < \infty$$

$\Rightarrow F' \in L^1[a,b] \,/\, G(x) = \int_a^x F'(s)\,ds \Rightarrow G \in AC[a,b]$

$H(x) = \underbrace{F(x) - G(x)}_{\in AC[a,b]} \qquad H'(x) = F'(x) - G'(x) = F'(x) - F'(x) = 0$ a.e

$\Rightarrow H$ is constant

$$H(x) = H(a) \Rightarrow \quad F(x) - G(x) = F(a) - G(a)$$

$$F(x) - \int_a^x F'(s)\,ds = F(a) - 0$$

$$F(x) - F(a) = \int_a^x F'(s)\,ds$$

$f \in L^1[a,b]$ $\qquad F(x) = \int_a^x f(s)\,ds \qquad h > 0$

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(s)\,ds = \frac{1}{m([x,x+h])}\int_{[x,x+h]} f(s)\,ds$$

$h \to 0$ $\qquad$ Lebesgue diff!

$\downarrow$

$f(x)$ $\qquad$ a.e