Differentiation & Integration

1. Fundamental Theorem of calculus
   a. BV functions (differentiable a.e, derivative integrable)
   b. Absolutely continuous functions

2. Differentiation of measures
   a. Signed measures
   b. Lebesgue - Radon - Nikodym theorem

Functions of bounded variation

Definition : Let $f : [a,b] \to \mathbb{R}$ f is said to be of bounded variation if

$$T_a^b(f) = \sup_P \left\{ \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| : P = \{t_i\}_{i=0}^{n} \right\} < \infty$$

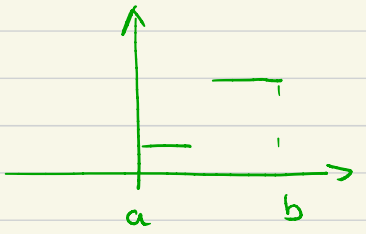
partition of $[a,b]$

Notation : $f \in BV[a,b]$

Theorem : $f \in BV[a,b]$ iff f is the difference of 2 monotone real valued functions on $[a,b]$.

Theorem : Let $f : [a,b] \to \mathbb{R}$ be an increasing function. Then f is differentiable a.e $x \in [a,b]$. The derivative f' is measurable and $f' \geq 0$ a.e. Moreover

$$\int_a^b f'(x)dx \leq f(b) - f(a)$$



Absolutely continuous functions

Definition: A function $F : [a,b] \to \mathbb{R}$ is absolutely continuous if $\forall \varepsilon > 0$ $\exists \delta > 0$ so that

$$\sum_{k=1}^{N} |F(b_k) - F(a_k)| < \varepsilon \quad \text{whenever} \quad \sum_{k=1}^{N} (b_k - a_k) < \delta$$

and $\{(a_k, b_k)\}_{k=1}^{N}$ are disjoint intervals in $[a,b]$.

Remarks: ① If F is absolutely continuous then F is continuous.

② If F is absolutely continuous in $[a,b]$ then $F \in BV[a,b]$.

③ If f is absolutely continuous on $[a,b]$ then so is $T_a^x(f)$.

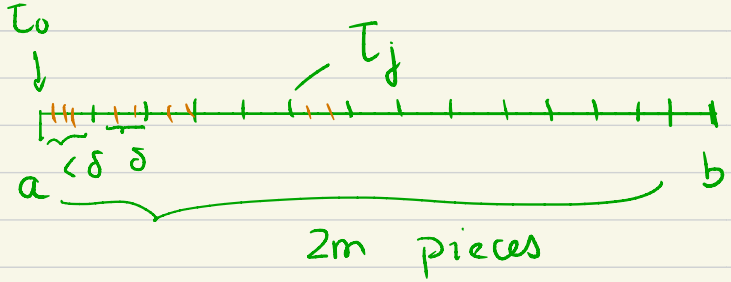
④ If $f \in L^1[a,b]$ & $F(x) = \int_a^x f(t)\,dt$ then F is absolutely continuous.

Q: What can we conclude from ④?

(2) If F is absolutely continuous in $[a,b]$ then $F \in BV[a,b]$.

(AC) $\forall \varepsilon > 0 \quad \exists \delta > 0$ so that

$$\sum_{k=1}^{N} |F(b_k) - F(a_k)| < \varepsilon \quad \text{whenever} \quad \sum_{k=1}^{N} (b_k - a_k) < \delta \quad (*)$$



$\varepsilon = 1 \qquad \delta = \frac{b-a}{n} \qquad P = \{t_i\}_{i=0}^{n}$

$\varepsilon = 1 \qquad \exists\, \delta = \frac{b-a}{m}$ s.t. $(*)$ holds $\qquad m \in \mathbb{N}$

$\tau_0 = a \qquad \tau_j = a + \frac{b-a}{2m} j \qquad j = 1 \ldots 2m$

$P_0 = \{\tau_j\}_{j=0}^{2m} \qquad P = \{t_i\}_{i=0}^{n} \qquad P' = P \cup P_0 = \{t'_k\}_{k=0}^{l}$

$\sum_{i=1}^{n} |F(t_i) - F(t_{i-1})| \le \sum_{i=1}^{l} |F(t'_i) - F(t'_{i-1})|$ (l part of $[a,b]$)

$\le \sum_{j=1}^{2m} \sum_{\tau_{j-1} < t'_i \le \tau_j} |F(t'_i) - F(t'_{i-1})|$ (t'_k part of $[a,b]$)

$$\sum_{\tau_{j-1} < t_i' \le \tau_j} |F(t_i') - F(t_{i-1}')| \le 1 \quad \text{by AC since}$$

$$\sum_{\tau_{j-1} < t_i' \le \tau_j} (t_i' - t_{i-1}') \le \tau_j - \tau_{j-1} = \frac{b-a}{2m}$$

$$\sum_{i=1}^{n} |F(t_i) - F(t_{i-1})| \le 2m \quad \Rightarrow \quad F \in BV[a,b]$$

③ If f is absolutely continuous on $[a,b]$ then so is $T_a^x(f)$ (↗)

$$T_a^x(f) = \sup \left\{ \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| : P = \{t_i\}_{i=0}^{n} \text{ part of } [a,x] \right\}$$

Remark: $a \le x < y < b$

Claim: $T_a^y(f) = T_a^x(f) + T_x^y(f)$ (please check)

Since f is AC $[a,b]$ $\quad \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \{(a_i,b_i)\}_{i=1}^{n}$ disjoint

(*) $\sum_{i=1}^{n} (b_i - a_i) < \delta \qquad \sum_{i=1}^{n} |f(b_i) - f(a_i)| < \varepsilon/2$

$$\sum_{i=1}^{n} T_a^{b_i}(f) - T_a^{a_i}(f) = \sum_{i=1}^{n} T_{a_i}^{b_i}(f)$$

for each i choose P_i partition $[a_i, b_i]$ $\quad P_i = \{t_\ell^i\}_{\ell=0}^{n_i}$ s.t

$$T_{a_i}^{b_i}(f) \le \sum_{\ell=1}^{n_i} |f(t_\ell^i) - f(t_{\ell-1}^i)| + \varepsilon/2n$$

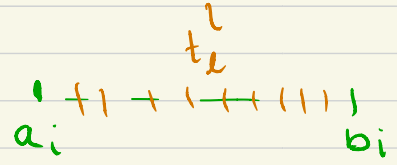
$$\sum_{i=1}^{n} T_{a_i}^{b_i}(f) \le \sum_{i=1}^{n}\left(\sum_{\ell=1}^{n_i} |f(t_\ell^i) - f(t_{\ell-1}^i)| + \varepsilon/2n\right)$$

$$\le \sum_{i=1}^{n} \left(\sum_{\ell=1}^{n_i} |f(t_\ell^i) - f(t_{\ell-1}^i)| \right) + \varepsilon/2$$

$$\{(t_{\ell-1}^i, t_\ell^i)\}_{\ell=1}^{n_i}, \quad 1 \le i \le n$$

$$\sum_{i=1}^{n}\left(\sum_{\ell=1}^{n_i}(t_\ell^i - t_{\ell-1}^i)\right) < \delta \quad \text{bec} \quad \sum(b_i - a_i) < \delta$$

$$\downarrow \text{ by } (\star)$$



AC

$$\sum_{i=1}^{n} T_{a_i}^{b_i}(f) \le \sum_{i=1}^{n} \left(\sum_{\ell=1}^{n_i} |f(t_\ell^i) - f(t_{\ell-1}^i)| \right) + \varepsilon/2 \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$