R problems

Due December 9 -11

Problem 1 (Prelim 2005) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and that there exists $k \in \mathbb{N}$ such that for every $y \in \mathbb{R}$ there are at most k distinct $x \in \mathbb{R}$ with $f(x) = y$. Prove that the derivative of f, $f'(x)$ exists for a.e. $x \in \mathbb{R}$.

Problem 2 (Prelim 2007) Let X be a compact metric space, and μ be a finite positive Borel measure on X. Suppose that $\mu(\{x\}) = 0$ for every $x \in X$. Prove that for every $\epsilon > 0$ there is a $\delta > 0$ such that, if E is any Borel subset of X having diameter less than δ, then $\mu(E) < \epsilon$.

Problem 3 (Prelim 2008) Let (X, d) be a metric space. A map $f : X \to X$ is said to be an isometry of (X, d) if $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Show that if (X, d) is a compact metric space and $f : X \to X$ is an isometry, then f is surjective.

Problem 4 (Prelim 2009) Let (X, d) be a compact space. A function $F : X \to \mathbb{R}$ is called upper semi- continuous if for every $x \in X$ and every $\epsilon > 0$ there is an open set U containing x such that $f(y) < f(x) + \epsilon$ for all $y \in U$.

- Show that f is upper semi-continuous if and only if $\{x : f(x) < a\}$ is open in X for every $a \in \mathbb{R}$.
- Show that if f is upper semi-continuous, there exists $K \in \mathbb{R}$ such that $f(x) < K$ for all $x \in X$.
- Show that if f is upper semi-continuous, then

$$f(x) = \inf\{g(x) : g \in C(X) \text{ and } g(y) > f(y) \text{ for all } y \in X\}.$$

Problem 5 (Prelim 2017) A function f is said to be convex on an interval $[a, b]$ if for all $x, y \in [a, b]$ and $0 \leq \lambda \leq 1$ we get

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

For a convex function $f : [a, b] \to \mathbb{R}$, prove the following:

- If $a < s < t < u < b$ then

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

- The left-hand and right-hand derivatives of $f$exist at every point of (a, b), and f must be continuous on (a, b).

Problem 6 (Prelim 2018) Let μ be a regular Borel measure on $\mathbb{R}^n$ such that $\mu(B) < \infty$ for any ball $B \subset \mathbb{R}^n$. Let E be a Borel set with $\mu(E) > 0$ such that

$$\limsup_{r \to 0} \frac{\mu(B(x, 2r)}{\mu(B(x, r)} < \infty \text{ for all } x \in E.$$

Show that there exists a closed set $E_0 \subset E$ with $0 < \mu(E_0) < \infty$ and constants $r_0 > 0$ and $M \geq 1$ such that for all $x \in E_0$ and $r \in (0, r_0)$

$$\frac{\mu(B(x, 2r)}{\mu(B(x, r)} \leq M.$$