HOMEWORK 8

**TI** problems due Wednesday November 25 at 10pm PT

Problems from Folland: Chapter 2, section 5: problems 46 and 48.

**Problem 1:** Let $f$ be a measurable finite-valued function on [0,1], and supposed that $h(x,y) = |f(x) - f(y)|$ is integrable in $[0,1] \times [0,1]$. Show that $f(x)$ is integrable in $[0,1]$.

**Problem 2:** Suppose $f$ is integrable on $\mathbb{R}^d$. For each $\alpha > 0$, let $E_\alpha = \{x : |f(x)| > \alpha\}$. Prove that

$$\int_{\mathbb{R}^d} |f(x)|\, dx = \int_0^\infty m(E_\alpha)\, d\alpha.$$

**Problem 3:** Suppose $f$ are $g$ are measurable functions on $\mathbb{R}^d$.

1. Prove that $f(x-y)g(y)$ is measurable in $\mathbb{R}^{2d}$.
2. If $f$ are $g$ are integrable on $\mathbb{R}^d$, then $f(x-y)g(y)$ is integrable in $\mathbb{R}^{2d}$.
3. Recall the definition of the convolution of $f$ and $g$ given by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\, dy.$$

   Show that $f * g$ is well defined for *a.e.* $x$ (that is $f(x-y)g(y)$ is integrable on $\mathbb{R}^d$ for *a.e.* $x$).

4. Show that $f * g$ is integrable whenever $f$ and $g$ are integrable, and that

$$\|f * g\|_{L^1(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)}.$$