R problems

Due Friday November 6

Problem 1: Let $f : [a, b] \to \mathbb{R}$ $(a, b \in \mathbb{R})$ be a bounded function. Prove that

- If f is Riemann integrable, then f is Lebesgue measurable and Lebesgue integrable. Moreover $\displaystyle\int_a^b f(x)\,dx = \int_{[a.b]} f\,dm$. Here m denotes the Lebesgue measure.
- f is Riemann integrable if and only if $\{x \in [a, b] : f$ is discontinuous at $x\}$ has Lebesgue measure 0.

Problem 2: Prove **Tchebychev inequality**. Let $(X, \mathcal{M}, \mu)$ be a complete measure space. Suppose that $f \in L^1(\mu)$. If $\alpha > 0$ then

$$\mu\{x \in X : f(x) > \alpha\} \leq \frac{1}{\alpha} \int f\,d\mu.$$

Problem 3: Let $(X, \mathcal{M}, \mu)$ be a complete measure space. Suppose $f : X \to [0, \infty]$ is measurable. Let

$$E_k = \{x \in X : f(x) > 2^k\} \qquad F_k = \{x \in X : 2^k < f(x) \leq 2^{k+1}\}.$$

Verify that if f is finite almost everywhere, then

$$\bigcup_{k=-\infty}^{\infty} F_k = \{x \in X : f(x) > 0\}.$$

Prove that the following are equivalent

1. $f \in L^1(\mu)$
2. $\displaystyle\sum_{k=-\infty}^{\infty} 2^k \mu(F_k) < \infty$
3. $\displaystyle\sum_{k=-\infty}^{\infty} 2^k \mu(E_k) < \infty$