HOMEWORK 7

TI problems due Wednesday November 18 at 10pm PT

Problem 1: (10 points) Suppose F is a closed set in $\mathbb{R}$, whose complement has finite measure, and let $\delta(x)$ denote the distance from x to F, that is

$$\delta(x) = d(x, F) = \inf\{|x - y| : \ y \in F\}.$$

Consider

$$I(x) = \int_{\mathbb{R}} \frac{\delta(y)}{|x - y|^2}\, dy.$$

1. Prove that δ is continuous, by showing that it satisfies the Lipschitz condition

$$|\delta(x) - \delta(y)| \leq |x - y|.$$

2. Show that $I(x) = \infty$ for each $x \notin F$.

3. Show that $I(x) < \infty$ for $a.e.$ $x \in F$. This may be surprising in view of the fact that the Lipschitz condition cancels only one power of $|x - y|$ in the integrand of I. [Hint: Investigate $\int_F I(x)\, dx$.]

Let $(X, \mathcal{M}, \mu)$ be a complete measure space. Let $\{f_n\}$ be a sequence of real valued measurable functions.

- The sequence $\{f_n\}$ is **Cauchy in measure** if for every $\epsilon > 0$

$$\mu(\{x : |f_n(x) - f_m(x)| \geq \epsilon\}) \to 0 \text{ as } m, n \to \infty.$$

- The sequence $\{f_n\}$ **converges in measure** to f if for every $\epsilon > 0$

$$\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \to 0 \text{ as } n \to \infty.$$

Problems from Folland: Chapter 2, section 4: problems 34, and 42.