R problem

Due Friday October 30

Problem 1: Integrability of f in $\mathbb{R}$ does not necessarily imply the convergence of $f(x)$ to 0 as $x \to \infty$.

1. There exists a continuous non-negative function f on $\mathbb{R}$ so that f is integrable on $\mathbb{R}$ but $\limsup_{x\to\infty} f(x) = \infty$.
2. However if we assume that f is uniformly continuous on $\mathbb{R}$ and integrable, then $\limsup_{|x|\to\infty} f(x) = 0$.

Problem 2: (Prelim 2000) Show in detail that

$$\int_0^1 \sum_{k=1}^{\infty} \frac{x^k \cos(2^k \pi x)}{k}\, dx = \sum_{k=1}^{\infty} \int_0^1 \frac{x^k \cos(2^k \pi x)}{k}\, dx.$$