HOMEWORK 3

TI problems due Wednesday October 21 at 10pm PT

Exercises from Folland:

Chapter 1: Section 3: problem 9.
Chapter 1: Section 5: problem 30 and 31.

Problem 1. Suppose $E \subset \mathbb{R}^d$ and $\mathcal{O}_n$ is the open set

$$\mathcal{O}_n = \left\{ x \in \mathbb{R}^d : d(x, E) < \frac{1}{n} \right\}.$$

Show:

1. If E is compact, then $m(E) = \lim_{n\to\infty} m(\mathcal{O}_n)$.
2. However, this may be false for E closed and unbounded; or E open and bounded. Provide examples.

Problem 2: The Borel-Cantelli Lemma Suppose $\{E_k\}_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Let

$$E = \limsup_{k\to\infty}(E_k) = \left\{ x \in \mathbb{R}^d : x \in E_k \text{ for infinitely many } \ k \right\}.$$

1. Show that E is (Lebesgue) measurable.
2. Show that $m(E) = 0$.

Hint: Write $E = \cap_{n=1}^{\infty} \cup_{k\geq n} E_k$.