HOMEWORK 2

TI problems due Wednesday October 14 at 10pm PT

Problem 1. Properties of the exterior measure m_* in $\mathbb{R}^d$. Recall for $E \subset \mathbb{R}^d$

$$m_*(E) = \inf\{\sum_j |Q_j| : E \subset \cup_j Q_j : \ Q_j \text{ closed cubes } \}.$$

For a cube $Q \subset \mathbb{R}^d$, $|Q|$ denotes the volume of Q.

1. Prove that the exterior (outer) measure m_* of a closed cube in $\mathbb{R}^d$ is its volume.
2. Prove that if $E \subset \mathbb{R}^d$ is a countable union of closed cubes such that their interiors are disjoint, that is
$$E = \cup_{i=1}^{\infty} Q_i \text{ with } \operatorname{int} Q_i \cap \operatorname{int} Q_j = \emptyset$$
then
$$m_*(E) = \sum_{j=1}^{\infty} m_*(Q_j).$$

Problem 2. Prove that if $E \subset \mathbb{R}^d$ and $E = E_1 \cup E_2$, with $d(E_1, E_2) > 0$ then

$$m_*(E) = m_*(E_1) + m_*(E_2).$$

Problem 3. Let F be a subset of $[0, 1]$ constructed in the same manner as the Cantor ternary set except that each of the intervals removed at the nth step has length $\alpha 3^{-n}$ with $0 < \alpha < 1$. Then F is closed, F^c is dense in $[0, 1]$ and $m_* F = 1 - \alpha$. Such set F is called a *generalized Cantor set.*

Problem 4. Let (X, ρ) be a non-empty complete metric space. Let $f : X \to X$ be such that there exists $\lambda \in (0, 1)$ satisfying

$$\rho(f(x), f(y)) \leq \lambda \rho(x, y) \ \text{ for all } \ x, y \in X.$$

Prove that there exists a unique point $u \in X$ such that $f(u) = u$. (Hint: let $x \in X$, and consider the sequence $x, \ f(x), \ f(f(x)), ...$). This result is known as Banach's fixed-point theorem. It implies several existence theorems in the theory of partial differential equations.

Problem 5 (Prelim 2001) Let (X, d) be a metric space and $K \subset X$ be compact. Let $\{G_i\}_{i \in I}$ be an open cover of K. Show that there exits $\epsilon > 0$ such that every subset S of K of diameter $< \epsilon$ is contained in one of the sets G_i (i.e. that there exists $i = i(S)$ such that $S \subset G_i$).