Theorem: Let $f : [a,b] \to \mathbb{R}$ be an increasing function. Then f is differentiable a.e $x \in [a,b]$. The derivative f' is measurable and $f' \geq 0$ a.e. Moreover

$$\int_a^b f'(x)\,dx \leq f(b) - f(a) \qquad (*)$$

Recall we have proved that

$$D^+ f(x) = D_+ f(x) = D^- f(x) = D_- f(x) \quad \text{m a.e } x \in [a,b]$$

We need to show that $m\{x \in [a,b] : D^+ f(x) = +\infty\} = 0$

$\Downarrow$ $(*)$

Recall

$$D^+ f(x) = \limsup_{k \to 0} \frac{f(x+k) - f(x)}{k}$$

$$m\{x \in [a,b] : D^+ f(x) > u\}$$

for $y \in \overbrace{\{D^+ f(x) > u\} \cap (a,b)}^{E}$ $\quad \forall n \in \mathbb{N} \quad \exists\, k_{y,\frac{1}{n}} < \frac{1}{n}$

$[y, y + k_{y,\frac{1}{n}}] \subset (a,b)$

$f(y + k_{y,\frac{1}{n}}) - f(y) > u\, k_{y,\frac{1}{n}}$ $\qquad \mathcal{B} = \{[y, y + k_{y,\frac{1}{n}}]\}_{\substack{y \in E \\ n \geq 1}}$

Given $\varepsilon > 0$

$I_1 \dots I_m$ $\qquad I_j = [y_j, y_j + k_j] \subset (a,b)$

disjoint

$$m\left(\{D^+ f(x) > u\} \cap (a,b) \setminus \bigcup_{j=1}^{m} I_j\right) < \varepsilon \qquad \Big\} (**)$$

$$\sum_{j=1}^{m} \left(f(y_j + k_j) - f(y_j)\right) > u \sum_{j=1}^{m} k_j$$

$$u\left(m\left(\{D^+ f(x) > u\} \cap (a,b)\right) - \varepsilon\right) \leq f(b) - f(a)$$

$$m\{D^+ f(x) > u\} \cap (a,b) - \varepsilon \leq \frac{f(b) - f(a)}{u}$$

$$m\{D^+ f(x) > u\} \cap (a,b) - \varepsilon \le \frac{f(b)-f(a)}{u}$$

$\forall \varepsilon > 0$

$$m\{D^+ f(x) > u\} \cap (a,b) \le \frac{f(b)-f(a)}{u}$$

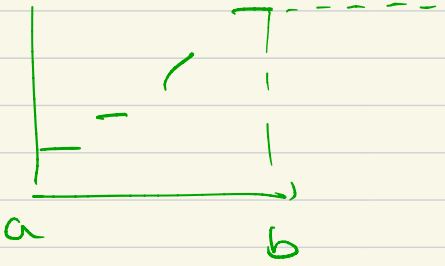
$$m\{D^+ f(x) = +\infty\} \le \frac{f(b)-f(a)}{u}$$

$u \to \infty$

Use Fatou to show (*)

$$\lim_{n \to 0} \frac{f(x+h)-f(x)}{h} = f'(x) \quad \text{a.e. finite}$$

$h = \frac{1}{n}$ set $f(x) = f(b)$ if $x > b$



$$g_n(x) = \frac{f(x+1/n)-f(x)}{1/n} = n\left(f\left(x+\frac{1}{n}\right)-f(x)\right) \ge 0 \text{ bec } f\uparrow$$

measurable

$$\lim_{n\to\infty} g_n(x) = f'(x) \qquad \text{a.e}$$

FATOU

$$\int_a^b f'(x)\,dx \le \liminf_{n\to\infty} \int_a^b g_n(x)\,dx$$

$$\int_a^b \frac{f(x+1/n) - f(x)}{1/n}\,dx = n\left(\int_a^b f(x+\tfrac{1}{n})\,dx - \int_a^b f(x)\,dx\right)$$

$$= n\left(\int_{a+\frac{1}{n}}^{b+1/n} f(y)\,dy - \int_a^b f(x)\,dx\right)$$

$$= n\left(\int_b^{b+1/n} f(y)\,dy - \int_a^{a+1/n} f(y)\,dy\right)$$

$$f(b) - f(a) \le n\left(\int_b^{b+1/n} f(b)\,dy - \int_a^{a+\frac{1}{n}} f(a)\,dy\right)$$

$\int_b^{b+1/n} f(b)\,dy = \frac{1}{n} f(b)$

$\int_a^{a+\frac{1}{n}} f(a)\,dy = \frac{1}{n} f(a)$

f ↑ $f(y) \ge f(a)$