Theorem: Let $f : [a,b] \to \mathbb{R}$ be an increasing function. Then f is differentiable a.e $x \in [a,b]$. The derivative f' is measurable and $f' \geq 0$ a.e. Moreover

$$\int_a^b f'(x)\,dx \leq f(b) - f(a)$$

Difference quotients

$$D^+ f(x) = \limsup_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

$$D_+ f(x) = \liminf_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

$$D^+ f(x) \geq D_+ f(x)$$

$$D^- f(x) = \limsup_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$$

$$D_- f(x) = \liminf_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$$

$$D^- f(x) \geq D_- f(x)$$

f is differentiable at x if

$$D^+ f(x) = D_+ f(x) = D^- f(x) = D_- f(x) = f'(x) \quad \text{finite}$$

Proof: To show the theorem we show that the set where 2 of these limits are different has measure 0.

$$E = \{ x \in (a,b) : D^+ f(x) > D_- f(x) \}$$

let $u, v \in \mathbb{Q}$

$$D^+ f(x) = \limsup_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \quad \text{is measurable}$$

$$D_- f(x) = \liminf_{h \to 0^-} \frac{f(x+h) - f(x)}{h} \quad \text{is measurable}$$

$$E_{u,v} = \{ x \in (a,b) : D^+ f(x) > u > v > D_- f(x) \}$$

Goal show $mE = 0$, $E = \bigcup_{u,v \in \mathbb{Q}} E_{u,v}$.

$E_{u,v}$ measurable, to show $m(E_{u,v}) = 0$

Given $\varepsilon > 0$ $\exists O$ open $E_{u,v} \subset O$ & $mO < mE_{u,v} + \varepsilon$

$mE_{u,v} \leq b - a < \infty$

if $x \in E_{u,v} \subset O$ $D_- f(x) = \liminf_{\tilde{h} \to 0^-} \frac{f(x+\tilde{h}) - f(x)}{\tilde{h}} = \sup_{\delta > 0} \inf_{\tilde{h} \in (-\delta, 0)} \frac{f(x+\tilde{h}) - f(x)}{\tilde{h}}$

$\therefore$ $= \sup_{\delta > 0} \inf_{\tilde{h} \in (0, \delta)} \frac{f(x-\tilde{h}) - f(x)}{-\tilde{h}} < v$ (*)

$\exists h_0 > 0 \quad (x - 2h_0, x + 2h_0) \subset O \qquad \forall n \in \mathbb{N}$

$\exists\ h_{x,\frac{1}{n}} \in (0, \min\{\frac{1}{n}, h_0\}) \quad \text{s.t} \quad [x - h_{x,\frac{1}{n}}, x] \subset O$

$$D_- f(x) = \sup_{\delta > 0} \inf_{\tilde{h} \in (0,\delta)} \frac{f(x-\tilde{h}) - f(x)}{-\tilde{h}} < v \Rightarrow \left| \frac{f(x - h_{x,\frac{1}{n}}) - f(x)}{-h_{x,1/n}} < v \right|$$

$$\circledast \quad \boxed{f(x) - f(x - h_{x,\frac{1}{n}}) < v\, h_{x,\frac{1}{n}}}$$

$I_{x,n} = [x - h_{x,\frac{1}{n}}, x] \qquad \mathcal{B} = \{I_{x,n}\}_{\substack{x \in E_{u,v} \\ n \in \mathbb{N}}}$ ← fine cover of $E_{u,v}$

$\forall \varepsilon > 0 \quad \exists$ finite disjoint subcollection $\{I_1, \dots I_m\} \subset \mathcal{B}$

$$m(E_{u,v} \setminus \bigcup_{i=1}^{m} I_i) < \varepsilon \qquad I_i = [x_i - h_i, x_i] \leftarrow$$

$$\sum_{i=1}^{m} f(x_i) - f(x_i - h_i) < v \sum_{i=1}^{m} h_i < v\, m O < v\,(m(E_{u,v}) + \varepsilon)$$

let $y \in E_{u,v} \cap \bigcup_{i=1}^{m} (x_i - h_i, x_i)$ disjoint.

$$\limsup_{k \to 0^+} \frac{f(y+k) - f(y)}{k} > u$$

$\forall \delta > 0 \quad \exists k_{y,\delta} \in (0, \delta) \quad (y, y + k_{y,\delta}) \subset (x_j - h_j, x_j)$ for some $j = 1, \dots m$

$$f(y + k_{y,\delta}) - f(y) > u k_{y,\delta}$$

$$\mathcal{B}' = \left\{ [y, y + k_{y,\frac{1}{n}}] \right\}_{\substack{y \in E_{u,v} \cap \bigcup_{j=1}^{m} (x_j - h_j, x_j) \\ n \in \mathbb{N}}}$$

is a fine cover $E_{u,v} \cap \bigcup_{j=1}^{m} (x_j - h_j, x_j)$

$\exists \left\{ [y_l, y_k + k_l] \right\}_{l=1}^{n} \subset \mathcal{B}$ s.t

$$m\left(E_{u,v} \cap \bigcup_{j=1}^{m} (x_j - h_j, x_j) \setminus \bigcup_{l=1}^{n} [y_l, y_l + k_l]\right) < \varepsilon$$

$$\sum_{l=1}^{n} |f(y_l + k_l) - f(y_l)| > u \sum_{l=1}^{n} k_l$$

$$m\left(E_{u,v} \cap \bigcup_{j=1}^{m} (x_j - h_j, x_j)\right) = m\left(E_{u,v} \cap \bigcup_{j=1}^{m} (x_j - h_j, x_j) \cap \bigcup_{l=1}^{n} [y_l, y_l + k_l]\right)$$

$$+ \underbrace{m\left(E_{u,v} \cap \bigcup_{j=1}^{m} (x_j - h_j, x_j) \setminus \bigcup_{l=1}^{m} [y_l, y_l + k_l]\right)}_{< \varepsilon}$$

$$\leq m\left(\bigcup_{l=1}^{n} [y_l, y_l + k_l]\right) + \varepsilon = \sum_{l=1}^{n} k_l + \varepsilon$$

$$m(E_{u,v}) = m\left(E_{u,v} \cap \bigcup_{j=1}^{m} (x_j - h_j, x_j)\right) + \underbrace{m\left(E_{u,v} \setminus \bigcup_{j=1}^{m} (x_j - h_j, x_j)\right)}_{< \varepsilon}$$

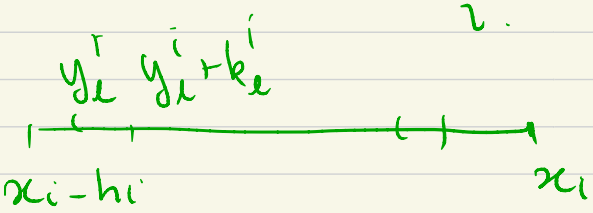
$$\boxed{\sum_{l=1}^{n} k_l \geq m(E_{u,v}) - 2\varepsilon}$$

(1) $\sum_{i=1}^{m} \left(f(x_i) - f(x_i - h_i) \right) < \nu \left(m E_{u,v} + \varepsilon \right)$

(2) $\sum_{k=1}^{n} \left(f(y_k + k_k) - f(y_k) \right) > u \left(m E_{u,v} - 2\varepsilon \right)$ ⊛

Recall $[y_k, y_k + k_k]$ disjoint inside $(x_i - h_i, x_i)$ for some i.

Call $[y_k^i, y_k^i + k_k^i] \subset (x_i - h_i, x_i)$



$$\sum_{k} \left(f(y_k^i + k_k^i) - f(y_k^i) \right) \leq f(x_i) - f(x_i - h_i)$$

↑ because $f \nearrow$

(3) $\sum_{k=1}^{n} \left(f(y_k + k_k) - f(y_k) \right) \leq \sum_{i=1}^{m} \left(f(x_i) - f(x_i - h_i) \right)$

$\forall \varepsilon > 0$

(1) & (2) (&) (3) $\quad u(m E_{u,v} - 2\varepsilon) < \nu(m E_{u,v} + \varepsilon)$

$\Rightarrow m E_{u,v} = 0$

We are missing:

$$m\{x : D^+ f(x) = +\infty\} = 0$$

$$\int_a^b f'(s)\,ds \leq f(b) - f(a)$$