Definition : let $f : [a,b] \to \mathbb{R}$ f is said to be of bounded variation if

$$(\star) = \sup_{P} \left\{ \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| : P = \{t_i\}_{i=0}^{n} \right\} < \infty$$

P partition of $[a,b]$

Notation : $f \in BV[a,b]$

Examples of functions of bounded variation :

① $f : [a,b] \to \mathbb{R}$ monotone bounded $(\star) \leq f(b) - f(a)$

② $C^1[a,b]$

③ Lipschitz ($\exists L > 0$ $|f(x) - f(y)| \leq L|x-y|$

$(\star) \leq L(b-a)$

④ Cantor function f

↑ monotone and bounded / continuous

a.e constant ; $f'(x) = 0$ a.e $\int_0^1 f'(x)\,dx = 0 \neq 1$

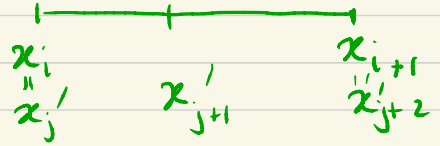
$f(1) - f(0) = 1$

P' refinement of P

Remark: If $P \subset P'$ 2 partitions of $[a,b]$ $\quad P = \{x_i\}_{i=0}^{n}$ $\quad P' = \{x_i'\}_{i=0}^{n+k}$

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^{n+k} |f(x_i') - f(x_{i-1}')| \quad \text{because}$$

by the $\Delta \leq$



$$|f(x_{i+1}) - f(x_i)| \leq |f(x_j') - f(x_{j+1}')| + |f(x_{j+1}') - f(x_{j+2}')|$$

Notation: $f: [a,b] \longrightarrow \mathbb{R}$ $\quad x \in [a,b]$ define

$$T_a^x(f) = \sup_{\substack{P \text{ partition} \\ \text{of } [a,x]}} \left\{ \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| : P = \{t_i\}_{i=0}^{n} \right\}$$

Total variation of f in $[a,x]$

$$P_a^x(f) = \sup_{\substack{P \text{ partition} \\ \text{of } [a,x]}} \left\{ \sum_{i=1}^{n} (f(t_i) - f(t_{i-1}))^+ : P = \{t_i\}_{i=0}^{n} \right\}$$



$$N_a^x(f) = \sup_{\substack{P \text{ partition} \\ \text{of } [a,x]}} \left\{ \sum_{i=1}^{n} (f(t_i) - f(t_{i-1}))^- : P = \{t_i\}_{i=0}^{n} \right\}$$

Recall $\quad a^{\pm} = \max\{0, \pm a\}$ $\quad a^+ - a^- = a$ $\quad \& \quad |a| = a^+ + a^-$

$\&\quad (a+b)^{\pm} \leq a^{\pm} + b^{\pm}$

Properties : For $x \in [a,b]$

(1) $f(x) - f(a) = P_a^x(f) - N_a^x(f)$.

(2) $T_a^x(f) = P_a^x(f) + N_a^x(f)$.

(3) $P_a^x(f)$, $N_a^x(f)$ are increasing functions.

$x < y$ $\quad [a,x] \subset [a,y]$, P partition $[a,y]$

$\widetilde{P} = P \cup \{x\} \cap [a,x]$ gives a partition of $[a,x]$

$$\sum_{\widetilde{P}} \leq \sum_{P \cup \{x\}}$$

Remark (1) & (2) are true if we use 1 partition at the time / approx $P_a^x(f)$ & $N_a^x(f)$ by sums for approp partition take a common refinement.

Given $\varepsilon > 0$ $\exists P = \{t_i\}_{i=0}^{n}$ $P' = \{t'_i\}_{i=1}^{m}$

$$\left| P_a^x(f) - \sum_{i=1}^{n} (f(t_i) - f(t_{i-1}))^+ \right| < \varepsilon/2$$

$$\left| N_a^x(f) - \sum_{i=1}^{m} (f(t'_i) - f(t'_{i-1}))^- \right| < \varepsilon/2$$

$P'' = P \cup P' = \{t''_i\}_{i=0}^{l}$ then $P \subset P''$

$$\sum_{i=0}^{l} (f(t''_i) - f(t''_{i-1}))^+ \le P_a^x(f) \le \sum_{i=1}^{n} (f(t_i) - f(t_{i-1}))^+ + \varepsilon/2$$

$$\le \sum_{i=1}^{l} (f(t''_i) - f(t''_{i-1}))^+ + \varepsilon/2$$

$$\sum_{i=0}^{l} (f(t''_i) - f(t''_{i-1}))^- \le N_a^x(f) \le \sum_{i=1}^{l} (f(t'_i) - f(t'_{i-1}))^- + \varepsilon/2$$

$$|f(t''_i) - f(t''_{i-1})| = (f(t''_i) - f(t''_{i-1}))^+ + (f(t''_i) - f(t''_{i-1}))^-$$

$$\underbrace{\sum_{i=1}^{\ell} |f(t_i'') - f(t_{i-1}'')|}_{(*)} \leq N_a^x(f) + P_a^x(f) \leq (*) + \varepsilon$$

sup on partitions

$$N_a^x(f) + P_a^x(f) \leq T_a^x(f) + \varepsilon$$

$$\overline{T}_a^x(f) \leq N_a^x(f) + P_a^x(f)$$

$\varepsilon \to 0$ $\qquad \overline{T}_a^x(f) = N_a^x(f) + P_a^x(f)$

Theorem: $f \in BV[a,b]$ iff f is the difference of 2 monotone real valued functions on $[a,b]$.

Pf $\Rightarrow$ | $x \in [a,b]$ $\quad g(x) = P_a^x(f) \nearrow \quad h(x) = N_a^x(f) \nearrow$

$$f(x) - f(a) = P_a^x(f) - N_a^x(f)$$

$$f(x) = [g(x) + f(a)] - h(x)$$

$\Leftarrow$ | $f = g - h$ $\quad g, h$ increasing $\quad P = \{t_i\}_{i=0}^n$ partition $[a,b]$

$$\sum_{i=1}^n |f(t_i) - f(t_{i-1})| \le \sum_{i=1}^n |g(t_i) - g(t_{i-1})| + \sum_{i=1}^n |h(t_i) - h(t_{i-1})|$$

$$T_a^b(f) \le g(b) - g(a) + h(b) - h(a)$$

$$\le N_a^b(f) + P_a^b(f)$$

Theorem: Let $f : [a,b] \to \mathbb{R}$ be an increasing function. Then f is differentiable a.e $x \in [a,b]$. The derivative f' is measurable and $f' \geq 0$ a.e. Moreover

$$\int_a^b f'(x)\,dx \leq f(b) - f(a)$$

Find f increasing, continuous such that

$$\int_a^b f'(x)\,dx \lneq f(b) - f(a)$$

} f Cantor function satisfies this

Corollary: If $f \in BV[a,b]$ then f' exists a.e $x \in [a,b]$

Moreover $$\int_a^b |f'(s)|\,ds \leq T_a^b(f)$$

Proof: $f = g - h$, $g, h \nearrow$; $g', h' \geq 0$ $\quad f' = g' - h' \quad |f'| \leq g' + h'$

$$\int_a^b |f'(s)|\,ds \le \int_a^b g'(s)\,ds + \int_a^b h'(s)\,ds$$

$$\le g(b) - g(a) + (h(b) - h(a))$$

$N_a^a(f) = 0$

$$g(x) = P_a^x(f) + f(a) \qquad h(x) = N_a^x(f)$$

$$\le P_a^b(f) + N_a^b(f) = T_a^b(f)$$

Difference quotients

$$D^+ f(x) = \limsup_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

$$D_+ f(x) = \liminf_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

$$D^+ f(x) \geq D_+ f(x)$$

$$D^- f(x) = \limsup_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$$

$$D_- f(x) = \liminf_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$$

$$D^- f(x) \geq D_- f(x)$$

f is differentiable at x if

$$D^+ f(x) = D_+ f(x) = D^- f(x) = D_- f(x) = f'(x) \quad \text{finite}$$

Proof: To show the theorem we show that the set where 2 of these limits are different has measure 0.