Lebesgue differentiation theorem

Theorem : If $f \in L^1(\mathbb{R}^n)$ then

$$(*) \qquad \lim_{\substack{m_B \to 0 \\ x \in B \\ B \text{ ball}}} \frac{1}{m(B)} \int_B f(y)\,dy = f(x) \quad \text{for } m\text{-a.e } x.$$

in fact

$$(**) \qquad \lim_{\substack{m_B \to 0 \\ x \in B \\ B \text{ ball}}} \frac{1}{m(B)} \int_B |f(y) - f(x)|\,dy = 0 \quad \text{for } m \text{ a.e } x.$$

(**) holds at the Lebesgue points for f.

Pf,

$$E_\alpha = \left\{ x : \limsup_{\substack{m(B) \to 0 \\ x \in B \\ B \text{ ball}}} \frac{1}{m(B)} \int_B |f(y) - f(x)|\,dy > 2\alpha \right\} \qquad \alpha > 0$$

Claim: $m(E_\alpha) = 0$

Fix $\alpha > 0$ $\quad \forall \varepsilon > 0 \quad \exists g_\varepsilon \in C_c(\mathbb{R}^n)$ s.t. $\quad \|f - g_\varepsilon\|_{L^1(\mathbb{R}^n)} < \varepsilon$

$(**)$ holds for g $\forall x \in \mathbb{R}^n$

$$\frac{1}{m(B)} \int_B |f(y) - f(x)|\,dy \le \underbrace{\frac{1}{m(B)} \int_B |f(y) - g(y)|\,dy}_{\le M(|f-g|)(x)}$$

$$+ \frac{1}{m(B)} \int_B |g(y) - g(x)|\,dy + \underbrace{\frac{1}{m(B)} \int_B |g(x) - f(x)|\,dy}_{|f(x) - g(x)|}$$

$\mathrm{diam}\, B \to 0$, $x \in B$, B ball $\Rightarrow$ 0

$$\frac{1}{m(B)} \int_B |f(y) - f(x)|\,dy \le M(|f-g|)(x) + |(f-g)(x)| + \frac{1}{m(B)} \int_B |g(y) - g(x)|\,dy$$

$$2\alpha < \limsup_{\substack{mB \to 0 \\ x \in B}} \frac{1}{m(B)} \int_B |f(y) - f(x)|\,dy \leq M(f-g)(x) + |f-g|(x)$$

$$F_\alpha = \{x : M(f-g)(x) > \alpha\}, \quad G_\alpha = \{x : |f-g|(x) > \alpha\}$$

either $M(f-g)(x) > \alpha \implies x \in F_\alpha$

or $|f-g|(x) > \alpha \implies x \in G_\alpha$

$$\fint_B |f(y)|\,dy = \frac{1}{m(B)} \int_B |f(y)|$$

$E_\alpha \subset F_\alpha \cup G_\alpha$, $\quad m(E_\alpha) \leq mF_\alpha + mG_\alpha$

Tchebychev

$$mG_\alpha \leq \frac{1}{\alpha} \|f-g\|_{L^1(\mathbb{R}^n)}$$

$$mF_\alpha \leq \frac{5^n}{\alpha} \|f-g\|_{L^1(\mathbb{R}^n)} \quad \text{(weak } L^1 \text{ ineq.)}$$

$\forall \varepsilon > 0$

$$m(E_\alpha) \leq \frac{1}{\alpha}(1+5^n)\|f-g\|_{L^1(\mathbb{R}^n)} < \frac{\varepsilon}{\alpha}(1+5^n)$$

$m(E_\alpha) = 0$ / Applying (x) to $|f|$ $\quad |f(x)| = \lim_{\substack{x \in B \\ m(B) \to 0}} \fint_B |f(y)| \leq Mf(x)$

m a.e

Corollary: If $f \in L^1_{loc}(\mathbb{R}^n)$ then

$$\lim_{\substack{m(B)\to 0 \\ x\in B \\ B \text{ ball}}} \frac{1}{m(B)} \int_B |f(y) - f(x)|\, dy = 0 \quad \text{for } m.a.e\ x$$

$f \in L^1_{loc}(\mathbb{R}^n)$ if for every Ball $\hat{B}$ $\int_{\hat{B}} |f| dy < \infty$

$f \in L^1_{loc}(\mathbb{R}^n)$ $\quad f\chi_B \in L^1(\mathbb{R}^n)$ $\quad \forall$ Ball

apply previous theorem $B_k \subset B_{k+1}$

and obtain the result.

Let $E \subset \mathbb{R}^n$ measurable $\quad \chi_E \in L^1_{loc}(\mathbb{R}^n)$

$$(*) \qquad \lim_{\substack{mB \to 0 \\ x \in B}} \frac{1}{mB} \int_B \chi_E(y)\,dy = \lim_{\substack{mB \to 0 \\ x \in B}} \frac{m(B \cap E)}{m(B)} = \chi_E(x) \quad \text{m.a.e}$$

$$= \begin{cases} 1 & \text{if } x \in \overline{E} \\ 0 & \text{if } x \notin E \end{cases}$$

If $\lim_{\substack{mB \to 0 \\ x \in B}} \frac{m(B \cap E)}{mB} = 1$ $\quad x$ is a Lebesgue density point of E

$\Downarrow$

$\forall \alpha \in (0,1) \quad \exists B \ni x \quad m(B \cap E) \geq \alpha\, mB$

Corollary : Suppose $E \subset \mathbb{R}^n$ measurable then

(i) m a.e $x \in \overline{E}$ is a density point of $\overline{E}$

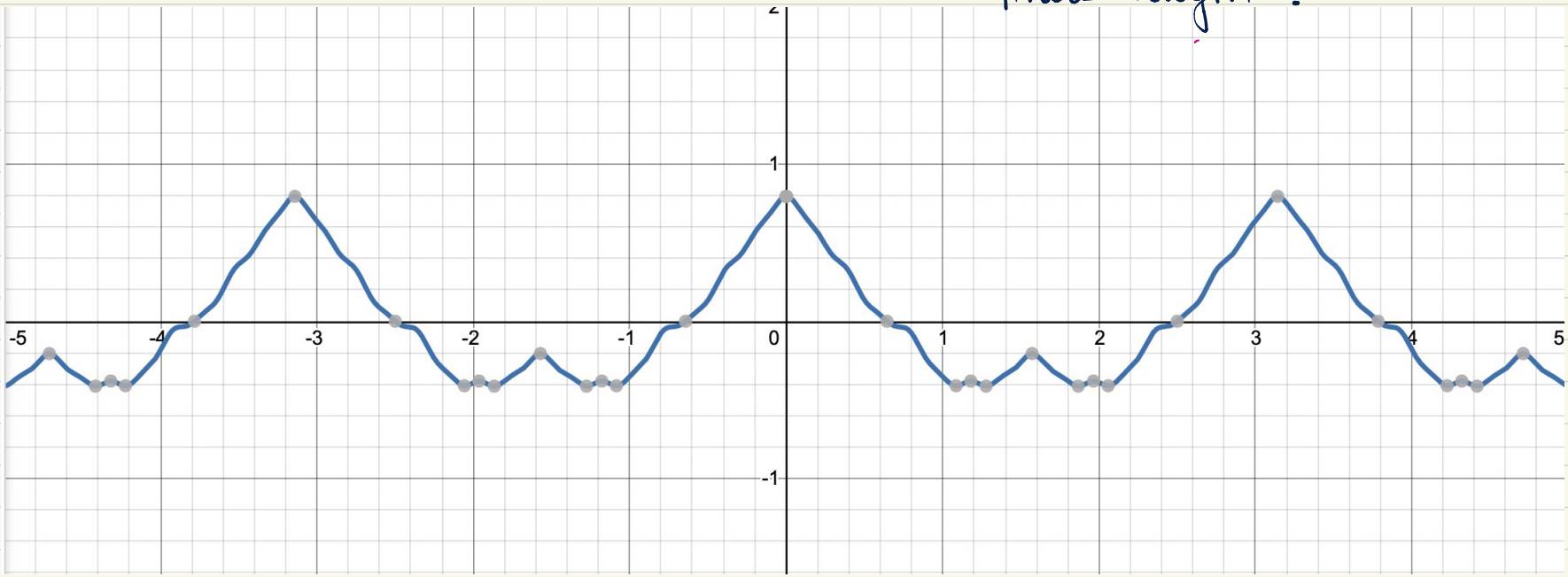
(ii) m a.e $x \notin \overline{E}$ is not a density point of E

Functions of bounded variation

Q: Are there conditions on functions defined on $\mathbb{R}$ which guarantee that f is differentiable a.e?

$$f(x) = \sum_{k=1}^{\infty} \frac{\cos(2^k x)}{2^k \sqrt{k}}$$

Does the curve γ parametrized by $(t, f(t))$ $t \in [-2, 2]$ have finite length?



$$\sum_{k=1}^{5} \frac{\cos(2^k x)}{2^k \sqrt{k}}$$

Geometric motivation: Let γ be a planar curve parameterized by $z(t) = (x(t), y(t))$ for $t \in [a,b]$.

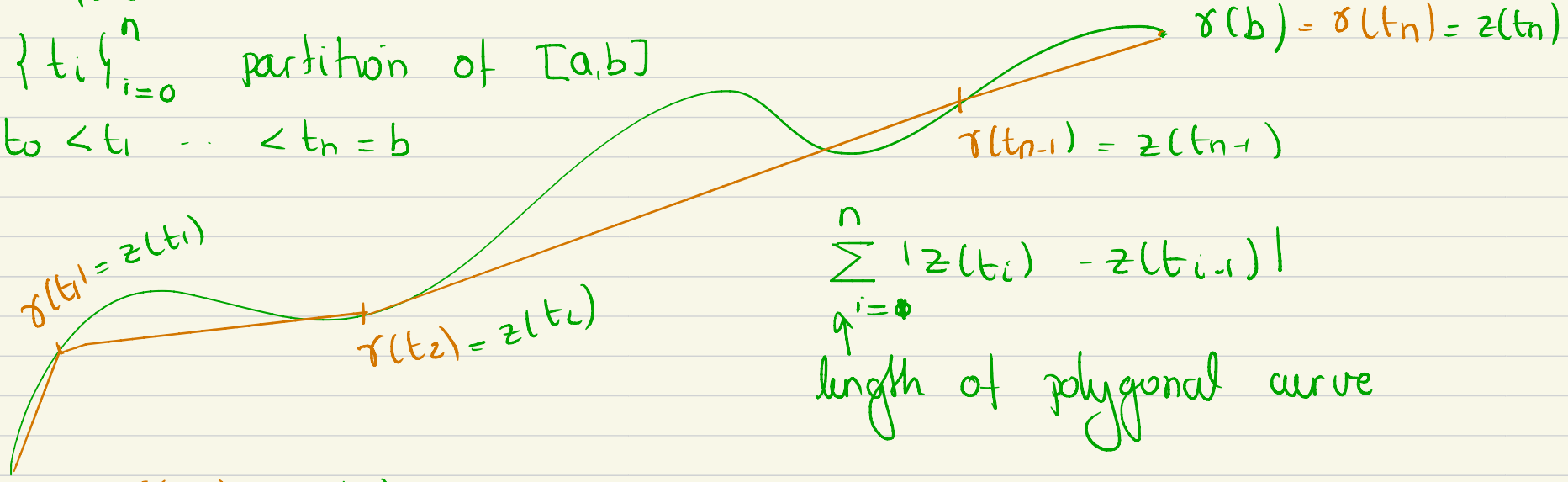
(1) How do you know if γ has finite length?

(2) How do you compute the length?

$\gamma : [a,b] \longrightarrow \mathbb{R}^2$

$P = \{t_i\}_{i=0}^{n}$ partition of $[a,b]$

$a = t_0 < t_1 \cdots < t_n = b$



$$\sum_{i=0}^{n} |z(t_i) - z(t_{i-1})|$$

length of polygonal curve

$$L(\gamma) = \sup\left\{ \sum_{i=0}^{n} |z(t_i) - z(t_{i-1})| : P = \{t_i\}_{i=0}^{n} \right\}$$

P: partition of $[a,b]$

$$\sum_{i=1}^{n} |z(t_i) - z(t_{i-1})| = \sum_{i=1}^{n} \sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2}$$

$$\overset{?}{=} \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

Definition : let $f : [a,b] \to \mathbb{R}$ f is said to be of bounded variation if

$$\sup_{P} \left\{ \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| : P = \{t_i\}_{i=0}^{n} \right\} < \infty$$

partition of $[a,b]$

Notation : $f \in BV[a,b]$

Theorem : A curve γ parameterized by $z(t) = (x(t), y(t))$ $a \le t \le b$ has finite length iff $x \in BV[a,b]$ & $y \in BV[a,b]$

$$\sum_{i=1}^{n} |x(t_i) - x(t_{i-1})| \,, \quad \sum_{i=1}^{n} |y(t_i) - y(t_{i-1})| \le \sum_{i=1}^{n} |z(t_i) - z(t_{i-1})|$$

$\Downarrow$

finite length $\Longrightarrow$ $x, y \in BV[a,b]$

$P = \{t_i\}_{i=0}^{n}$

$$\sum_{i=1}^{n} |z(t_i) - z(t_{i-1})| \le \sum_{i=1}^{n} |x(t_i) - x(t_{i-1})| + \sum_{i=1}^{n} |y(t_i) - y(t_{i-1})|$$