Differentiation of an integral

Question: If $f \in L^1_{loc}(\mathbb{R}^n)$ is it true that

$$(*) \qquad \lim_{\substack{m_B \to 0 \\ x \in B \\ B \text{ ball}}} \frac{1}{m(B)} \int_B f(y)\,dy = f(x) \quad \text{m. a.e } x \in \mathbb{R}^n ?$$

$$\left(\lim_{\substack{m_B \to 0 \\ x \in B \text{ ball}}} \frac{1}{m(B)} \int_B |f(y) - f(x)|\,dy = 0 \right)$$

↑ if f cont. at x

Claim: if f is continuous at x $(*)$ holds.

$\forall \varepsilon > 0 \quad \exists r > 0$ s.t. if $|x - y| < r$ then $|f(x) - f(y)| < \varepsilon$

$$\left| \frac{1}{m(B)} \int_B f(y)\,dy - f(x) \right| = \left| \frac{1}{m(B)} \int_B (f(y) - f(x))\,dy \right|$$

$$\leq \frac{1}{m(B)} \int_B \underbrace{|f(y) - f(x)|}_{< \varepsilon}\,dy < \varepsilon$$

$B = B(z, r_B)$ $\quad r_B < r/4$, if $x, y \in B$ $\quad |x - y| \leq 2r_B < \frac{r}{2}$

The Hardy - Littlewood maximal function

If $f \in L^1(\mathbb{R}^n)$ the maximal function of f is defined by

$$f^*(x) = Mf(x) = \sup_{\substack{x \in B \\ B \text{ ball}}} \frac{1}{m(B)} \int_B |f(y)|\, dy$$

Theorem: Suppose $f \in L^1(\mathbb{R}^n)$ then

i) Mf is measurable

ii) $Mf(x) < \infty$ for m . a . e $x \in \mathbb{R}^n$

iii) Mf satisfies (a weak L^1 - inequality) $\forall \alpha > 0$

$$(*) \quad m\{x \in \mathbb{R}^n : Mf(x) > \alpha\} \leq \frac{5^n}{\alpha} \|f\|_{L^1(\mathbb{R}^n)}$$

Recall Tchebychev inequality

$$(**) \quad m\{x \in \mathbb{R}^n : |f(x)| > \alpha\} \leq \frac{1}{\alpha} \|f\|_{L^1(\mathbb{R}^n)}$$

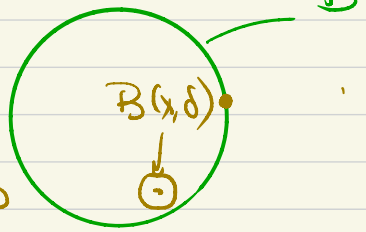
(we will show $Mf(x) \geq |f(x)|$ m a.e)

$$Mf(x) = \sup_{\substack{x \in B \\ B \text{ ball}}} \frac{1}{m(B)} \int_B |f(y)|\,dy$$

Claim : i)

$E_\alpha = \{ x \in \mathbb{R}^n : Mf(x) > \alpha \}$ is open, $x \in E_\alpha$ $\exists B \ni x$ (ball)

s.t $\frac{1}{m(B)} \int_B |f(y)|\,dy > \alpha$ (*)



if B is open $\forall z \in B$ $\exists\, B(z, \delta_z) \subset B$

apply to $z = x$ $\forall \bar{z} \in B(x, \delta_x) \subset B \Rightarrow Mf(\bar{z}) > \alpha$

$B(x, \delta_x) \subset E_\alpha \Rightarrow E_\alpha$ is open

if B is not open (*) is a strict inequality enlarge B a little to make open $B = \overline{B(x_0, r_0)}$ $\tilde{B} = B(x_0; (1+\varepsilon) r_0)$

s.t

$$\frac{1}{m(B)} \int_B |f(y)|\,dy = \frac{1}{(1+\varepsilon)^n\, m(B)} \int_{\tilde{B}} |f(y)|\,dy \geq \underbrace{\frac{1}{(1+\varepsilon)^n} \frac{1}{m(B)} \int_B |f|}_{> \alpha}$$

for ε small

Assume iii) $\{x \in \mathbb{R}^n : Mf(x) = \infty\} \subset \{Mf(x) > \alpha\}$ $\forall \alpha > 0$

$$\underbrace{m\{Mf = +\infty\}}_{} \leq m\{Mf > \alpha\} \leq \frac{5^n}{\alpha} \|f\|_{L^1(\mathbb{R}^n)}$$

letting $\alpha \to \infty$

$$m\{Mf = +\infty\} = 0$$

iii) Show $\underbrace{m(E_\alpha)}_{} \leq \frac{5^n}{\alpha} \|f\|_{L^1(\mathbb{R}^n)}$ ↙

$x \in E_\alpha$ $\exists B_x \ni x$ closed non-degenerate $\underbrace{\frac{1}{m(B_x)} \int_{B_x} |f(y)|\,dy}_{} > \alpha$

B_x has radius r_x

(*) $\alpha \underbrace{m(B_x)}_{} = \alpha \underbrace{\omega_n}_{\text{measure of } B_1} r_x^n < \underbrace{\int_{B_x} |f(y)|\,dy}_{} \leq \int_{\mathbb{R}^n} |f(y)|\,dy$

$$r_x \leq \left(\frac{1}{\alpha \omega_n} \|f\|_{L^1(\mathbb{R}^n)}\right)^{1/n} = C(n, \alpha, \|f\|_{L^1(\mathbb{R}^n)})$$

$$\mathcal{F} = \{ \overline{B_x} \text{ (closed non-deg)} : x \in E_\alpha : \frac{1}{m(B_x)} \int_{B_x} |f| \, dy > \alpha \}$$

$$\sup_{B_x \in \mathcal{F}} \operatorname{diam} B_x \leq c(n, \alpha, \|f\|_{L^1(\mathbb{R}^n)})$$

$\exists \mathcal{G} \subset \mathcal{F}$ countable disjoint $\mathcal{G} = \{B_i\}_{i=1}^{\infty}$

$$E_\alpha \subset \bigcup_{x \in E_\alpha} B_x \subset \bigcup_{B_i \in \mathcal{G}} \hat{B}_i$$

$$m(E_\alpha) \leq \sum_{B_i \in \mathcal{G}} m(\hat{B}_i) = \sum_{B_i \in \mathcal{G}} 5^n m(B_i)$$

$$\leq 5^n \sum_{B_i \in \mathcal{G}} \frac{1}{\alpha} \int_{B_i} |f(y)| \, dy = \frac{5^n}{\alpha} \int_{\bigcup B_i} |f(y)| \leq \frac{5^n}{\alpha} \|f\|_{L^1(\mathbb{R}^n)}$$

(B_i disjoint)

Lebesgue differentiation theorem

Theorem: If $f \in L^1(\mathbb{R}^n)$ then

$$(*) \quad \lim_{\substack{m_B \to 0 \\ x \in B \\ B \text{ ball}}} \frac{1}{m(B)} \int_B f(y)\,dy = f(x) \quad \text{for } m\text{-a.e } x.$$

in fact

$$(**) \quad \lim_{\substack{m_B \to 0 \\ x \in B \\ B \text{ ball}}} \frac{1}{m(B)} \int_B |f(y) - f(x)|\,dy = 0 \quad \text{for } m \text{ a.e } x.$$

(**) holds at the Lebesgue points for f.

Pf

$$E_\alpha = \left\{ x : \limsup_{\substack{m(B) \to 0 \\ x \in B \\ B \text{ ball}}} \frac{1}{m(B)} \int_B |f(y) - f(x)|\,dy > 2\alpha \right\} \quad \alpha > 0$$

Goal: Show that $m(E_\alpha) = 0$, let E be the set where (**) does not hold $E = \bigcup_n E_{1/n}$, if $m(E_{1/n}) = 0$ ✓

Fix $\alpha > 0$ $\quad \forall \varepsilon > 0 \quad \exists g_\varepsilon \in C_c(\mathbb{R}^n)$ s.t. $\quad \|f - g_\varepsilon\|_{L^1(\mathbb{R}^n)} < \varepsilon$

$(**)$ holds for g $\forall x \in \mathbb{R}^n$

$$\frac{1}{m(B)} \int_B |f(y) - f(x)|\,dy \le \underbrace{\frac{1}{m(B)} \int_B |f(y) - g(y)|\,dy}_{\le M(|f-g|)(x)}$$

$$+ \frac{1}{m(B)} \int_B |g(y) - g(x)|\,dy + \underbrace{\frac{1}{m(B)} \int_B |g(x) - f(x)|\,dy}_{|f(x) - g(x)|}$$

$\mathrm{diam}\, B \to 0$, $x \in B$, B ball $\Rightarrow$ 0

$$\frac{1}{m(B)} \int_B |f(y) - f(x)|\,dy \le M(|f-g|)(x) + |(f-g)(x)| + \frac{1}{m(B)} \int_B |g(y) - g(x)|\,dy$$