Theorem: Vitali's covering theorem

Let $\mathcal{F}$ be any collection of non-degenerate closed balls in $\mathbb{R}^n$ such that

$$D = \sup \{ \operatorname{diam} B : B \in \mathcal{F} \} < \infty$$

Then there exists a countable family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that

$$\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B \in \mathcal{G}} \hat{B}$$

Moreover if $B \in \mathcal{F}$ $\exists S \in \mathcal{G}$ s.t $B \cap S \neq \emptyset$ and $B \subset \hat{S}$.

Pf $\mathcal{F}_j = \{ B \in \mathcal{F} : \frac{D}{2^j} < \operatorname{diam} B \leq \frac{D}{2^{j-1}} \}$ $j \geq 1$

We define $\mathcal{G}_j \subset \mathcal{F}_j$ as follows

$$\mathcal{F}_j = \{ B \in \mathcal{F} : \frac{D}{2^j} < \operatorname{diam} B \le \frac{D}{2^{j-1}} \}$$

a) $\mathcal{G}_1$ a maximal disjoint collection in $\mathcal{F}_1$

b) Assume $\mathcal{G}_1 \dots \mathcal{G}_{k-1}$ have been selected; choose $\mathcal{G}_k$ to be a maximal disjoint subcollection

$$\{ B \in \mathcal{F}_k : B \cap B' = \emptyset \quad \forall B' \in \bigcup_{j=1}^{k-1} \mathcal{G}_j \}$$

Define $\mathcal{G} = \bigcup_{j=1}^{\infty} \mathcal{G}_j$ - is a collection of disjoint balls (by construction) + it is countable

if $B \in \mathcal{F}$, then there j s.t $B \in \mathcal{F}_j$ either $B \in \mathcal{G}_j$

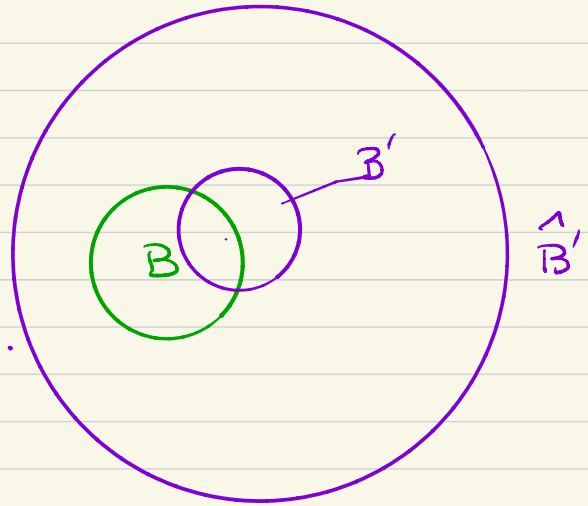
or not $B \notin \mathcal{G}_j$ $\exists B' \in \bigcup_{k=1}^{j} \mathcal{G}_k$ $B' \cap B \neq \emptyset$

$B \in \mathcal{F}_j$ $\frac{D}{2^j} < \operatorname{diam} B \le \frac{D}{2^{j-1}}$ $\quad$ $\downarrow$ $\operatorname{diam} B' > \frac{D}{2^j}$

$\operatorname{diam} \le 2 \operatorname{diam} B'$

$B \cap B' \neq \emptyset$ $\quad$ $\operatorname{diam} B \leq 2 \operatorname{diam} B'$

$\Rightarrow B \subset \hat{B}'$



Corollary: Assume that $\mathcal{F}$ is a fine cover of A by non-degenerate closed balls and

$$\sup\{\operatorname{diam} B : B \in \mathcal{F}\} < \infty$$

Then there exists a countable family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ s.t. for any finite subset $\{B_1, \dots B_m\} \subset \mathcal{F}$ we have

$$A \setminus \bigcup_{k=1}^{m} B_k \subset \bigcup_{B \in \mathcal{G} \setminus \{B_1, \dots B_m\}} \hat{B}$$

Pf: Let $\mathcal{G}$ be as in Vitali covering theorem, let $\{B_1, \dots B_m\} \subset \mathcal{F}$

If $A \subset \bigcup_{k=1}^{m} B_k$ ✓; if $x \in A \setminus \bigcup_{k=1}^{m} B_k$

Since $\mathcal{F}$ is a fine cover & $\bigcup_{k=1}^{m} B_k$ is closed there is $B \in \mathcal{F}$ s.t $x \in B$ $B \cap B_k = \emptyset$ $k = 1, \dots m$

$\exists S \in \mathcal{G}$ $\quad B \cap S \neq \phi \quad B \subset \hat{S}$ $\quad$ if $S \neq B_k$

$k = 1, \dots m$

$x \in B \subset \hat{S}$

$A \quad \bigcup_{k=1}^{m} B_k \subset \bigcup \hat{S}$

$S \in \mathcal{G} \setminus \{B_1 \dots B_m\}$

corollary: Let $E \subset \mathbb{R}^n$ be measurable s.t. $m(E) < \infty$. Let $\mathcal{B}$ be a collection of (non-trivial) closed balls covering E finely. Then given $\varepsilon > 0$ there exists a finite disjoint subcollection $\{B_1, \dots, B_m\} \subset \mathcal{B}$ s.t.

$$m\left(E \setminus \bigcup_{i=1}^{m} B_i\right) < \varepsilon$$

Pf: $mE < \infty$ $\exists$ O open $E \subset O$; $mO < \infty$ $(m(O \setminus E) < \varepsilon)$

$\widetilde{\mathcal{F}}_0 = \{B \in \mathcal{F}_e : \operatorname{diam} B \le 1\}$

since $\mathcal{F}$ fine cover of E so is $\widetilde{\mathcal{F}}_0$

$\mathcal{F}_0 = \{B \in \mathcal{F} : \operatorname{diam} B \le 1 \quad B \subset O\}$ fine cover of E

by corollary $\exists$ $\mathcal{C}_y$ countable disjoint subcollection of $\mathcal{F}_0$

$$E \subset \bigcup_{B \in \mathcal{C}_y} \hat{B} \qquad \mathcal{C}_y = \{B_i\}_{i=1}^{\infty}$$

$\{B_1, \ldots, B_m\} \subset \mathcal{G}$ $\qquad B_i \subset O$

(*) $\quad E \setminus \bigcup_{k=1}^{m} B_k \subset \bigcup_{B \in \mathcal{G} \setminus \{B_1 \ldots B_m\}} \hat{B}$

$$\sum_{i=1}^{\infty} m(\hat{B}_i) = \sum_{i=1}^{\infty} 5^n m(B_i) = 5^n \sum_{i=1}^{\infty} m(B_i) \quad \text{(disjoint)}$$

$$\leq 5^n \, m\left(\bigcup_{i=1}^{\infty} B_i\right) = 5^n \, m(O) < \infty$$

given $\varepsilon > 0$

$\exists m$ s.t $\quad \sum_{i=m+1}^{\infty} m(\hat{B}_i) < \varepsilon$

(*) $\Rightarrow \quad m\left(E \setminus \bigcup_{i=1}^{m} B_k\right) \leq m\left(\bigcup_{i=m+1}^{\infty} \hat{B}_i\right) \leq \sum_{i=m+1}^{\infty} m(\hat{B}_i) < \varepsilon$

Question: If $f \in L^1_{loc}(\mathbb{R}^n)$ what can you say about

$$\lim_{\substack{m(B)\to 0 \\ x\in B \\ B \text{ ball}}} \frac{1}{m(B)} \int_B f(y)\,dy \quad ?$$

$$\frac{1}{2r_0} \int_{x_0-r}^{x_0+r} f(y)\,dy \longrightarrow f(x_0) \quad \text{if } f \text{ is continuous}$$

$$r \to 0$$