The Fubini - Tonelli Theorem

Suppose that $(X, \mathcal{A}, \mu)$ & $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces

i) [Tonelli] If $f \in \mathcal{L}^+(X \times Y)$ then the functions

(*) $g(x) = \int f_x(y)\, d\nu(y)$ & $h(y) = \int f^y(x)\, d\mu(x)$ satisfy

$g \in \mathcal{L}^+(X)$ and $h \in \mathcal{L}^+(Y)$. Moreover

$$(*) \quad \int f\, d(\mu \times \nu) = \int \underbrace{\left(\int f(x,y)\, d\mu(x) \right)}_{h(y)} d\nu(y) = \int \underbrace{\left(\int f(x,y)\, d\nu(y) \right)}_{g(x)} d\mu(x)$$

ii) [Fubini] If $f \in L^1(\mu \times \nu)$ then $f_x \in L^1(\nu)$ μ-a.e $x \in X$ and $f^y \in L^1(\mu)$ ν-a.e $y \in Y$. The functions g and h in (*) satisfy $g \in L^1(\mu)$ and $h \in L^1(\nu)$. Furthermore (*) holds.

Pf ① Tonelli (*) & (**) hold χ_E $E \in \mathcal{A} \otimes \mathcal{N}$.

② Tonelli holds for simple functions

③ if $f \in \mathcal{L}^+(X \times Y)$ by 2.10 $\{f_n\}$ simple functions

(•) $0 \le f_n \le f_{n+1} \le f$ $f_n \to f$ pointwise

$((f_n)_x \le (f_{n+1})_x)$

By MCT

$$0 \le g_n(x) = \int (f_n)_x(y)\, d\nu(y) \le \int (f_{n+1})_x(y)\, d\nu(y) = g_{n+1}(x)$$

$$0 \le g_n(x) \le g_{n+1}(x) \le g(x) \quad g_n \to g(x) = \int f_x(y)\, d\nu(y)$$

↑ measurable

By MCT again

$$\int g(x)\, d\mu(x) = \lim_{n\to\infty} \int g_n(x)\, d\mu(x) = \lim_{n\to\infty} \int f_n\, d(\mu\times\nu)$$

$$\text{MCT } (\bullet) = \int f\, d(\mu\times\nu) < \infty \Rightarrow \int g(x)\, d\mu(x) < \infty$$

If $f \in L^+(X\times Y) \cap L^1(\mu\times\nu)$

$\Rightarrow g(x) < \infty$ μ-a.e x

$\Rightarrow f_x \in L^1(\nu)$ μ-a.e x

Fubini : $f \in L^1(\mu \times \nu)$ $f = f^+ - f^-$ $f^{\pm} \geq 0$

all conclusions of Fubini hold $f^{\pm}$ so by linearity they hold for f.

Differentiation & Integration

Goal: Formulate and prove the fundamental theorem of calculus in this setting

(1) Suppose that $f \in L^1[a,b]$ & $x \in [a,b]$ define

$$F(x) = \int_a^x f(t)\,dt = \int_{[a,x]} f(t)\,dt$$

What is the relationship between F' & f?

(2) Assume that $F : [a,b] \longrightarrow \mathbb{R}$, what are the minimal conditions that guarantee that F' exists a.e, $F' \in L^1[a,b]$ & $F(b) - F(a) = \int_a^b F'(t)\,dt$?

(3) How does this translate to functions in $L^1_{loc}(\mathbb{R}^n)$?

Covering theorems

If $B = \overline{B(x,r)} = \{y \in \mathbb{R}^n : |x-y| \le r\}$ we denote by

$$\hat{B} = 5B = \overline{B(x,5r)}$$

Definitions : (1) A collection $\mathcal{F}_c$ of closed balls in $\mathbb{R}^n$ is a cover of a set $A \subset \mathbb{R}^n$ if

$$A \subset \bigcup_{B \in \mathcal{F}_c} B$$

(2) $\mathcal{F}$ is a fine cover of A if $\mathcal{F}_c$ is a cover of A and

$$\inf \{ \operatorname{diam} B : x \in B : B \in \mathcal{F}_c \} = 0 \quad \text{for each } x \in A.$$

Examples : $A = B(x_0, r_0) \subset \mathbb{R}^n$ $\quad \mathcal{F}_c = \left\{ \overline{B(x, \tfrac{1}{2})} \right\}_{x \in A}$

$\hat{\mathcal{F}} = \{ B(x_i, \tfrac{1}{n}) \}_{n \in \mathbb{N}, \, x_i \in \mathbb{Q}^n \cap B(x_0, r_0)}$ fine cover

$\tilde{\mathcal{F}} = \{ B(x, \tfrac{1}{n}) \}_{\substack{x \in A \\ n \in \mathbb{N}}}$

Theorem: Vitali's covering theorem

Let $\mathcal{F}$ be any collection of non-degenerate closed balls in $\mathbb{R}^n$ such that

$$D = \sup \{ \operatorname{diam} B : B \in \mathcal{F} \} < \infty$$

Then there exists a countable family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that

$$\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B \in \mathcal{G}} \hat{B}$$

Moreover if $B \in \mathcal{F}$ $\exists S \in \mathcal{G}$ s.t $B \cap S \neq \emptyset$ and $B \subset \hat{S}$.

Remark: what happens if $D = +\infty$? it fails

↓ $\{B(0,n)\}_{n \in \mathbb{N}}$

$\mathcal{F} = \{B\}_{B \in \mathcal{F}^0}$

