Theorem: Suppose $(X, \mathcal{M}, \mu)$ & $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces. If $E \in \mathcal{M} \otimes \mathcal{N}$ then the functions $E_x = \{y \in Y : (x,y) \in E\}$

$$(*) \quad \begin{cases} X \longrightarrow [0,\infty] & Y \longrightarrow [0,\infty] \\ x \longrightarrow \nu(E_x) & y \longrightarrow \mu(E^y) \\ \mathcal{M}\text{-measurable} & \mathcal{N}\text{-measurable} \end{cases}$$

Moreover

$$(**) \quad \mu \times \nu(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y)$$

Pf: ① assume μ & ν are finite

$$\mathcal{C} = \{ E \in \mathcal{M} \otimes \mathcal{N} \text{ s.t } (*) \text{ & } (**) \text{ hold} \}$$

Goal: rectangles are in $\mathcal{C}$ & $\mathcal{C}$ is a σ-algebra

② μ & ν σ-finite

Pf: ① assume μ & ν are finite

$$\mathcal{C} = \{ E \in \mathcal{M} \otimes \mathcal{N} \text{ s.t } (*) \text{ & } (**) \text{ hold} \}$$

Claim 1: $\mathcal{H} = \{$ finite union of disjoint rectangles $\} \subset \mathcal{C}$

Claim 2: $\mathcal{C}$ is a σ-algebra

Claim 1 ⊕ Claim 2: since $\mathcal{M} \otimes \mathcal{N}$ σ-algebra generated by $\mathcal{R}$ then $\mathcal{M} \otimes \mathcal{N} \subset \mathcal{C}$. Since $\mathcal{C} \subset \mathcal{M} \otimes \mathcal{N}$ by definition then

$$\mathcal{C} = \mathcal{M} \otimes \mathcal{N}$$

Claim 1: $\mathcal{H} = \{$ finite union of disjoint rectangles $\} \subset \mathcal{C}$

$E = A \times B \qquad E_x = \{ y \in Y : (x,y) \in E \} = B$

$$\nu(E_x) = \nu((A\times B)_x) = \chi_A(x)\,\nu(B) \leftarrow \mathcal{M}\text{-meas.}$$

$$= \int \chi_{E_x}(y)\, d\nu(y) = \int (\chi_E)_x(y)\, d\nu(y)$$

$$= \int \chi_E(x,y)\, d\nu(y) = \int \chi_{A\times B}(x,y)\, d\nu(y)$$

$$= \int \chi_A(x)\,\chi_B(y)\, d\nu(y)$$

$$\mu(E^y) = \chi_B(y)\,\mu(A) \leftarrow \mathcal{N}\text{-meas.}$$

$$\mu\times\nu(A\times B) = \mu(A)\,\nu(B) = \int \chi_A(x)\,\nu(B)\, d\mu(x)$$

$$= \int \nu(E_x)\, d\mu(x)$$

rectangles $\subset \mathcal{C}$

why is $\mathcal{A} \subset \mathcal{C}$?

$\mathcal{A}$ = finite disjoint unions

$A \times B \cup C \times D$ — disjoint

$$\underbrace{E}$$

$$\mu \times \nu (A \times B \cup C \times D) = \mu \times \nu (A \times B) + \mu \times \nu (C \times D)$$

$$= \int \nu((A \times B)_x)\, d\mu(x) + \int \nu((C \times D)_x)\, d\mu(x)$$

$$= \int \chi_A(x)\, \nu(B)\, d\mu(x) + \int \chi_C(x)\, \nu(D)\, d\mu(x)$$

$$= \int \underbrace{(\chi_A(x)\, \nu(B) + \chi_C(x)\, \nu(D))}\, d\mu(x)$$

$$\int \nu[\underbrace{(A \times B \cup C \times D)_x}]\, d\mu(x)$$

E_x

Definition: A monotone class $\mathcal{C}$ on X is a subset of 2^X which is closed under countable increasing unions and countable decreasing intersections.

Remarks: ① A σ-algebra is a monotone class

② The intersection of any family of monotone classes is a monotone class. Hence for $\mathcal{E} \subset 2^X$ there is a unique smallest monotone class containing $\mathcal{E}$ it is the monotone class generated by $\mathcal{E}$ (intersection of all monotone classes containing $\mathcal{E}$)

Technical lemma: If $\mathcal{A}$ is an algebra (i.e a closed collection under finite unions and complements) then the monotone class generated by $\mathcal{A}$ coincides with the σ-algebra generated by $\mathcal{A}$.

Claim 2': $\mathcal{C} = \{ E \in \mathcal{A} \otimes \mathcal{N}$ s.t (*) & (**) hold $\}$ is a monotone class

If $\{E_n\}_n \subset \mathcal{C}$, $E_n \subset E_{n+1}$, $E = \bigcup_{n=1}^{\infty} E_n \in \mathcal{C}$ (goal)

$f_n(y) = \mu(E_n^y)$ is $\mathcal{N}$- measurable, $E_n^y \subset E_{n+1}^y$

$0 \le f_n \le f_{n+1} \le \dots \le f$, $f(y) = \mu(E^y)$

$f = \lim_{n\to\infty} f_n = \sup_{n\ge 1} f_n$ is $\mathcal{N}$- measurable

by MCT

$$\lim_{n\to\infty} \int f_n \, d\nu(y) = \int f(y)\, d\nu(y) = \int \mu(E^y)\, d\nu(y)$$

$E_n \in \mathcal{C}$, $\int \mu(E_n^y)\, d\nu(y)$

$$\lim_{n\to\infty} \mu \times \nu(E_n) = \mu \times \nu(E) = \int \mu(E^y)\, d\nu(y)$$

If $\{E_n\} \subset \mathcal{C}$ $\quad E_{n+1} \subset E_n$ $\quad E = \bigcap_{n=1}^{\infty} E_n \in \mathcal{C}$ ← goal

$y \longmapsto \mu(E_1^y) \in L^1(\nu)$ because $\mu \times \nu(E_1) < \infty$

measurable

$$\int \mu(E_1^y)\, d\nu(y) = \mu \times \nu(E_1) \le \mu \times \nu(X \times Y) = \mu(X)\,\nu(Y) < \infty$$

$E_{n+1} \subset E_n \Rightarrow E_{n+1}^y \subset E_n^y$

$\mu(E_n^y) \le \mu(E_1^y)$ $\qquad \mu(E^y) = \inf_n \mu(E_n^y)$

LDCT ↓ $\qquad$ ↑ $\mathcal{A}$ - measurable

$$* \int \mu(E^y)\, d\nu(y) = \lim_{n\to\infty} \int \mu(E_n^y)\, d\nu(y) = \lim_{n\to\infty} \mu \times \nu(E_n) = \mu \times \nu(E)$$

MCT $\quad \mu(E_1^y) - \mu(E_n^y) \le \mu(E_1^y) - \mu(E_{n+1}^y)$

$$\lim_{n\to\infty} \int (\mu(E_1^y) - \mu(E_n^y))\, d\nu(y) = \int (\mu(E_1^y) - \mu(E^y))\, d\mu(y)$$

$$\int \mu(E_1^y)\, d\nu(y) - \lim_{n\to\infty} \int \mu(E_n^y)\, d\nu(y) \qquad \mu \times \nu(E_n)$$

② μ & ν σ-finite $\quad X = \bigcup_j X_j \quad X_j \subset X_{j+1} \quad \mu(X_j) < \infty$

$Y = \bigcup_j Y_j \quad Y_j \subset Y_{j+1} \quad \nu(Y_j) < \infty$

$X \times Y = \bigcup_j X_j \times Y_j$ because $(x,y) \in X \times Y$

$x \in X_j \quad y \in Y_k \Rightarrow x \in X_{j+k}, \; y \in Y_{j+k}$

$\Rightarrow (x,y) \in X_{j+k} \times Y_{j+k}$

If $E \in \mathcal{M} \otimes \mathcal{N}$ the previous argument ensures

$$\mu \times \nu (E \cap X_j \times Y_j) = \int \underbrace{\nu(E_x \cap Y_j)}_{\mathcal{M}\text{-measurable}} d\mu \llcorner X_j(x)$$

$$= \int_{X_j} \nu(E_x \cap Y_j)\, d\mu(x)$$

$$= \int_{Y_j} \mu(E^y \cap X_j)\, d\nu(y)$$

($\leftarrow$ $\mathcal{N}$-measurable)

The Fubini - Tonelli Theorem

Suppose that $(X, \mathcal{A}, \mu)$ & $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces

i) [Tonelli] If $f \in \mathcal{L}^+(X \times Y)$ then the functions

(*) $g(x) = \int f_x(y)\, d\nu(y)$ & $h(y) = \int f^y(x)\, d\mu(x)$ satisfy

$g \in \mathcal{L}^+(X)$ and $h \in \mathcal{L}^+(Y)$. Moreover

$$(*) \quad \int f\, d(\mu \times \nu) = \int \left(\int f(x,y)\, d\mu(x) \right) d\nu(y) = \int \left(\int f(x,y)\, d\nu(y) \right) d\mu(x)$$

ii) [Fubini] If $f \in L^1(\mu \times \nu)$ then $f_x \in L^1(\nu)$ μ-a.e $x \in X$ and $f^y \in L^1(\mu)$ ν-a.e $y \in Y$. The functions g and h in (*) satisfy $g \in L^1(\mu)$ and $h \in L^1(\nu)$. Furthermore (*) holds.