Product Measures

Let $(X, \mathcal{M}, \mu)$ & $(Y, \mathcal{N}, \nu)$ be (complete) measure spaces

Goal: construct a measure $\mu \times \nu$ on the σ-algebra $\mathcal{M} \otimes \mathcal{N}$

Let $\pi_X : X \times Y \to X$ & $\pi_Y : X \times Y \to Y$

$(x, y) \mapsto x$ $\qquad$ $(x, y) \mapsto y$

$\mathcal{M} \otimes \mathcal{N}$ = σ algebra generated by $\mathcal{E}$ where

$$\mathcal{E} = \{ \pi_X^{-1}(E) \}_{E \in \mathcal{M}} \cup \{ \pi_Y^{-1}(F) \}_{F \in \mathcal{N}}$$

$\mathcal{R} = \{ A \times B : A \in \mathcal{M}, B \in \mathcal{N} \}$ collection of rectangles

- $\phi \in \mathcal{R}$
- $(A \times B) \cap (E \times F) = A \cap E \times B \cap F$
- $(A \times B)^c = A^c \times B \cup A \times B^c \cup A^c \times B^c$

disjoint

$= A^c \times Y \cup A \times B^c$



$\mathcal{A} = \{$ finite union of disjoint rectangles $\} \neq \emptyset$

Then $\mathcal{A}$ is closed under finite unions and complements $\therefore$ $\mathcal{A}$ is an algebra

$\mathcal{M} \otimes \mathcal{N}$ is the σ-algebra generated by $\mathcal{A}$ (check)

If $A \times B = \bigcup_j A_j \times B_j$ disjoint union $A_j \in \mathcal{M}$ $B_j \in \mathcal{N}$

$x \in X$ $y \in Y$

$$\chi_A(x)\,\chi_B(y) = \chi_{A\times B}(x,y) = \sum_j \chi_{A_j \times B_j}(x,y) = \sum_j \chi_{A_j}(x)\,\chi_{B_j}(y)$$

integrate w.r.t x

$$\mu(A)\,\chi_B(y) = \int \sum_j \chi_{A_j}(x)\,\chi_{B_j}(y)\,d\mu(x)$$

$$= \sum_j \int \chi_{A_j}(x)\,\chi_{B_j}(y)\,d\mu(x)$$

$$= \sum_j \mu(A_j)\,\chi_{B_j}(y)$$

$$\mu(A)\,\chi_B(y) = \sum_j \mu(A_j)\chi_{B_j}(y)$$ integrate w.r.t y

use MCT

- $$\mu(A)\,\nu(B) = \sum_j \mu(A_j)\,\nu(B_j)$$ ⟵

If $E \in \mathcal{A}$ if $E = \bigcup_{j=1}^{n} A_j \times B_j$ disjoint

define $$\pi(E) = \sum_{i=1}^{n} \mu(A_i)\,\nu(B_i)$$ π is well defined

$$E = \bigcup_{j=1}^{n} A_j \times B_j = \bigcup_{i=1}^{m} C_i \times D_i = \bigcup_{j=1}^{n}\bigcup_{i=1}^{m} A_j \cap C_i \times B_j \cap D_i$$

disjoint union

Claim 1 : $\pi : \mathcal{A} \longrightarrow [0, \infty]$ satisfies

i) $\pi \phi = \phi$

ii) if $\{A_\ell\} \subset \mathcal{A}$ disjoint & $\bigcup A_\ell \in \mathcal{A}$

then $\pi(\bigcup A_\ell) = \sum_\ell \pi(A_\ell)$ (check)

Claim 2 : for $E \subset X \times Y$

$$\pi^*(E) = \inf \left\{ \sum_{j \geq 1} \pi(E_j) : E_j \in \mathcal{A} \quad E \subset \bigcup E_j \right\}$$

① π^* is an outer measure ($\pi^* \phi = 0$ & π^* monotone subadditive)

② $\pi^* \big|_{\mathcal{A}} = \pi$

③ if $E \in \mathcal{A} \Rightarrow E$ is π^* measurable (ie $\forall S \subset X \times Y$

$$\pi^*(S) = \pi^*(S \cap E) + \pi^*(S \cap E^c) \text{) (check)}$$

Since $\mathcal{M} \otimes \mathcal{N}$ is generated by $\mathcal{C}$, and all sets in $\mathcal{C}$ are π^*-measurable then all sets in $\mathcal{M} \otimes \mathcal{N}$ are π^*-measurable

(all measurable sets form a σ-algebra)

$$\mu \times \nu = \pi^* \Big|_{\mathcal{M} \otimes \mathcal{N}}$$

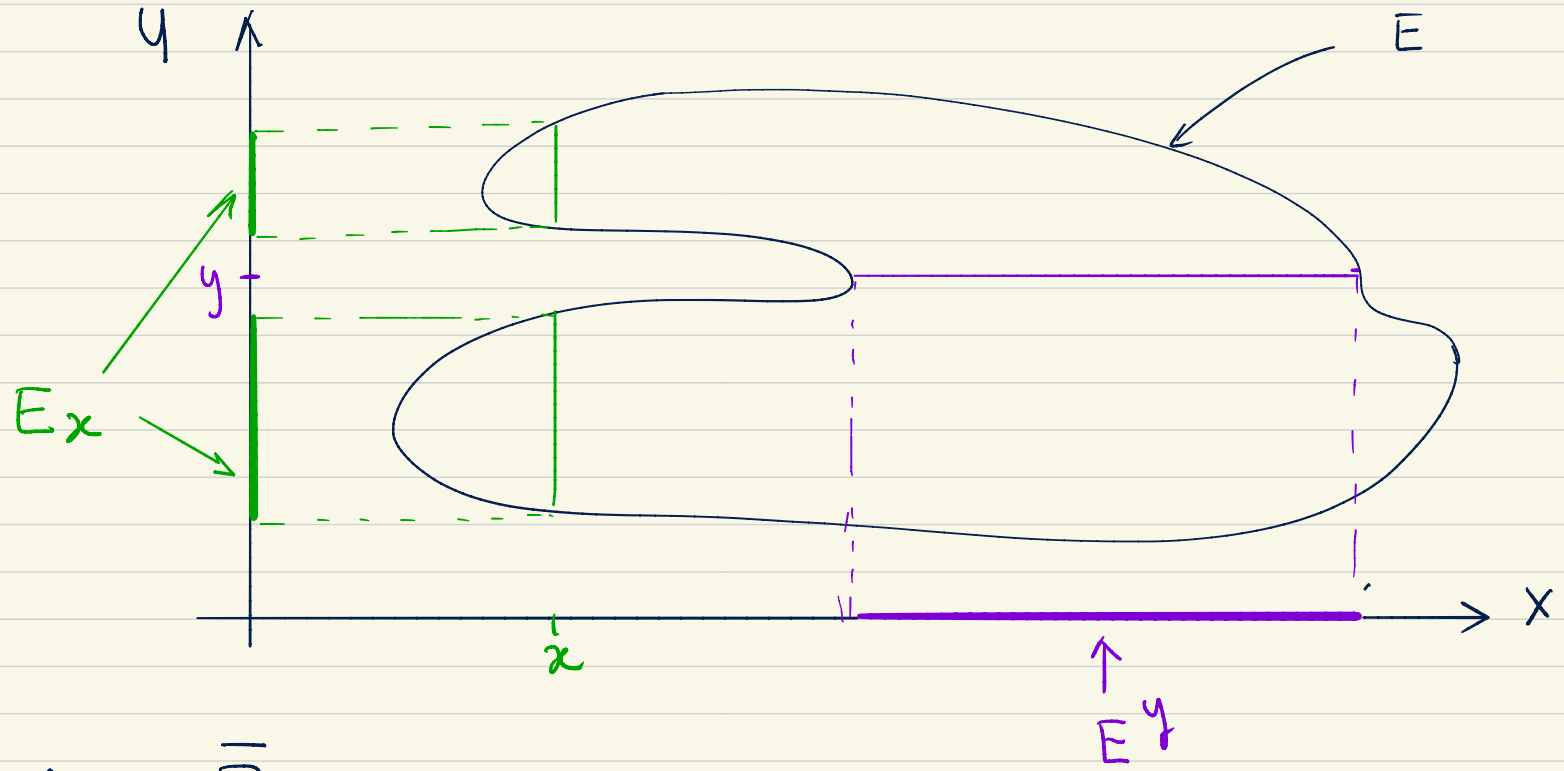
If μ & ν are σ-finite then $\mu \times \nu$ is σ-finite

In this case $\mu \times \nu$ is the unique extension of π to $\mathcal{M} \otimes \mathcal{N}$

If $E \subset X \times Y$ for $x \in X$ & $y \in Y$ define

$$E_x = \{ y \in Y : (x,y) \in E \} \quad x\text{-section}$$

$$E^y = \{ x \in X : (x,y) \in E \} \quad y\text{-section}$$



$f : X \times Y \longrightarrow \overline{\mathbb{R}}$

$f_x : Y \longrightarrow \overline{\mathbb{R}}$ $\quad f_x(y) = f(x,y)$ $\quad x$-section

$f^y : X \longrightarrow \overline{\mathbb{R}}$ $\quad f^y(x) = f(x,y)$ $\quad y$-section

Note if $E \subset X \times Y$ $\quad (\chi_E)_x = \chi_{E_x}$ $\quad (\chi_E)^y = \chi_{E^y}$

Proposition: i) If $E \in \mathcal{M} \otimes \mathcal{N}$ then $E_x \in \mathcal{N}$ $\forall x \in X$ and $E^y \in \mathcal{M}$ $\forall y \in Y$

ii) if $f : X \times Y \longrightarrow \overline{\mathbb{R}}$ $\mathcal{M} \otimes \mathcal{N}$ measurable then
f_x is $\mathcal{N}$ - measurable for every $x \in X$
f^y is $\mathcal{M}$ - measurable for every $y \in Y$. check

Goal: Relate

$$\int f \, d(\mu \times \nu) \quad \text{with} \quad \int \left(\int \underbrace{f(x,y)}_{f_x(y)} \, d\nu(y) \right) d\mu(x)$$

and

$$\int \left(\int f^y(x) \, d\mu(x) \right) d\nu(y)$$

Q: What is the key?

Theorem: Suppose $(X, \mathcal{M}, \mu)$ & $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces. If $E \in \mathcal{M} \otimes \mathcal{N}$ then the functions

$$(*) \quad \begin{cases} X \longrightarrow [0,\infty] & Y \longrightarrow [0,\infty] \\ x \longrightarrow \nu(E_x) & y \longrightarrow \mu(E^y) \\ \mathcal{M}\text{-measurable} & \mathcal{N}\text{-measurable} \end{cases}$$

Moreover

$$(**) \quad \mu \times \nu(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y)$$

$$= \int \left(\int \chi_{E_x}(y)\, d\nu(y) \right) d\mu(x)$$

$$= \int \left(\int (\chi_E)_x(y)\, d\nu(y) \right) d\mu(x)$$

$$\int \chi_E(x,y)\, d(\mu \times \nu) = \int \left(\int \chi_E(x,y)\, d\nu(y) \right) d\mu(x)$$