Egoroff's theorem: Let $(X, \mathcal{M}, \mu)$ be a complete measure space. Suppose $\mu(X) < \infty$, $\{f_n\}$ are measurable and $f_n \to f$ μ-a.e. Then given $\varepsilon > 0$ $\exists E_\varepsilon \subset X$ measurable s.t. $\mu(E_\varepsilon) < \varepsilon$ and $f_n \to f$ uniformly on E_ε^c

Pf: $Y = \{x \in X : f_n(x) \to f(x)\}$ $\quad \mu(Y^c) = 0$

$$E_{n,k} = \bigcup_{m=n}^{\infty} \{x \in Y : |f_m(x) - f(x)| \geq \frac{1}{2^k}\}$$

Fix k $\quad E_{n+1,k} \subset E_{n,k}$ $\quad \bigcap_{n=1}^{\infty} E_{n,k} = \emptyset$ because $\subset Y$ and $f_n(x) \to f(x)$ $n \to \infty$ in Y

$$0 = \mu\left(\bigcap_{n=1}^{\infty} E_{n,k}\right) = \lim_{n \to \infty} \mu(E_{n,k}) \text{ because}$$

$E_{1,k} \subset X$ & $\mu(E_{1,k}) \leq \mu(X) < \infty$

$$\lim_{n \to \infty} \mu(E_{n,k}) = 0$$

$\forall \varepsilon > 0 \quad \exists n_k \quad \mu(E_{n_k,k}) < \frac{\varepsilon}{2^k}$

$$E = \bigcup_{k=1}^{\infty} E_{n_k,k} \qquad \mu(E) \le \sum_{k=1}^{\infty} \mu(E_{n_k,k}) \le \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$$

$$E_{n_k,k} = \bigcup_{m=n_k}^{\infty} \{ x \in \Omega : |f_m(x) - f(x)| \ge \tfrac{1}{2^k} \}$$

Claim $f_n \to f$ uniformly on E^c.

If $x \notin E$ $\quad \forall k \quad x \notin E_{n_k,k} \quad \forall m \ge n_k$

$$|f_m(x) - f(x)| < 1/2^k$$

If $\eta > 0$ choosing k_0 s.t. $\frac{1}{2^{k_0}} < \eta$

Remarks: ① What happens when $\mu(X) = +\infty$?

② The hypothesis $\mu(X) < \infty$ can be relaxed (pb 40 Folland)

Pf $X = \mathbb{R}$ $\mu = m$ $f_n = \chi_{(n,\infty)}$ $f_n \to 0$ m.a.e

shows Egoroff fails if $m(\mathbb{R}) = +\infty$

$f_n = \chi_{(-n,n)}$ $f_n = \chi_{(n,n+1)}$

Lusin's Theorem: If $f : [a,b] \to \mathbb{R}$ Lebesgue measurable, for any $\varepsilon > 0$ there is $E_\varepsilon \subset [a,b]$ compact such that $\mu(E_\varepsilon^c) < \varepsilon$ and $f|_{E_\varepsilon}$ is continuous

(Prelim)

Problem 1: Let $L^1(\mathbb{R}^d)$ be the set of Lebesgue integrable functions in $\mathbb{R}^d$, let m be the Lebesgue measure in $\mathbb{R}^d$ and $\|f\|_1 = \int_{\mathbb{R}^d} |f|\, dm$. Let $\{f_n\}_{n\geq 1}$ be a sequence in $L^1(\mathbb{R}^d)$. Suppose that there exists $f \in L^1(\mathbb{R}^d)$ such that $f_n \to f$ m-a.e. and $\|f_n\|_1 \to \|f\|_1$.

1. Show that for all Lebesgue sets $E \subset \mathbb{R}^d$

$$\int_E |f_n|\, dm \to \int_E |f|\, dm \text{ as } n \to \infty.$$

2. Prove that

$$\|f_n - f\|_1 \to 0 \text{ as } n \to \infty.$$

3. Show that for every continuous function with compact support $\varphi \in C_c(\mathbb{R}^d)$

$$\lim_{n\to\infty} \int f_n \varphi\, dm = \int f \varphi\, dm$$

↳

support of φ = spt φ = $\overline{\{x \in \mathbb{R}^d : f(x) \neq 0\}}$