Theorem: A normed vector space is complete iff every absolutely convergent series in X converges in X.

Pf: $\Rightarrow$ ✓

$\Leftarrow$ $\{x_n\}_n$ Cauchy sequence, find a convergent subsequence is enough; choose $n_1 < n_2 < \dots < n_k$ s.t.

$$\|x_n - x_m\| \le \frac{1}{2^k} \qquad \forall n, m \ge n_k \qquad (\{x_n\} \text{ Cauchy})$$

$y_1 = x_{n_1}$; $y_2 = x_{n_2} - x_{n_1}$, $y_k = x_{n_k} - x_{n_{k-1}}$, ...

$$\sum_{k=1}^{l} y_k = x_{n_l}, \qquad \sum_{k=1}^{\infty} \|y_k\| = \sum_{k=2}^{\infty} \|x_{n_k} - x_{n_{k-1}}\| + \|x_{n_1}\|$$

$$\le \|x_{n_1}\| + \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} \le \|x_{n_1}\| + 2$$

↑ requires a vector space

$\sum \|y_n\|$ abs. convergent by hyp: $\sum_{k=1}^{l} y_k = x_{n_l} \xrightarrow[l\to\infty]{} x \in X$

thus $\{x_n\}$ Cauchy then it converges.

Corollary: Let $\{f_n\} \subset L^1(\mu)$ & assume that $f_n \to f$ in $L^1(\mu)$ then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ μ a.e.

Pf: $\{f_n\}$ is Cauchy, $\exists\ n_1 < n_2 < \dots < n_k$ s.t

$\forall n, m \geq n_k \qquad \|f_n - f_m\| < \frac{1}{2^k}$

$g_1 = f_{n_1} \qquad g_2 = f_{n_2} - f_{n_1} \qquad g_k = f_{n_k} - f_{n_{k-1}}$

$$\sum_{k=1}^{\ell} g_k = f_{n_\ell}$$

$$\sum_{k=1}^{\infty} \|g_k\| \leq \|f_{n_1}\| + 2 < \infty$$

abs. convergent

$$\sum_{k=1}^{\infty} \int |g_k| \, d\mu < \infty$$

then

$$f_{n_\ell} = \sum_{k=1}^{\ell} g_k \xrightarrow{L^1(\mu)} \sum_{k=1}^{\infty} g_k = g \in L^1(\mu) \quad \text{(by previous theorem)}$$

(need $f_{n_\ell} \Rightarrow g$ μ a.e & $f = g$ μ a.e)

↑ from last prop. class

want to show

$$\int |f-g|\,d\mu = 0$$

$f_{n_\ell} \to g$ μ.a.e $L^1(\mu)$

$f_{m_\ell} \to f$ in $L^1(\mu)$

$$\underbrace{|f_{n_\ell} - g|}_{\downarrow\ \mu.a.e\ 0} \le |f_{n_\ell}| + |g| = \left|\sum_{k=1}^{\ell} g_k\right| + |g|$$

$$\le 2 \underbrace{\sum_{j=1}^{\infty} |g_j|}_{G} = 2G$$

$$\int G\,d\mu = \int \sum_{j=1}^{\infty} |g_j| \underset{MCT}{=} \sum_{j=1}^{\infty} \int |g_j|\,d\mu < \infty$$

DCT

$$\lim_{\ell\to\infty} \int |f_{n_\ell} - g|\,d\mu = \int |f-g|\,d\mu = 0 \Rightarrow f = g \quad \mu.a.e$$

Theorem: If $f \in L^1(\mu)$; given $\varepsilon > 0$ there exists ϕ integrable and simple such that

$$\int |\phi - f| \, d\mu < \varepsilon$$

i.e. simple integrable functions are dense in $L^1(\mu)$.

Pf: $f = f^+ - f^-$ $\qquad f^\pm = \max\{0; \pm f\}$

Thm 2.10 $\quad 0 \le \phi_i^\pm \le \phi_{i+1}^\pm \le \dots \le f^\pm$

$\phi_i^\pm$: measurable / simple $\qquad |\xi_i| \le |f| \quad \xi_i \in L^1(\mu)$

$\phi_i^\pm \longrightarrow f^\pm \quad \mu\text{-a.e} \qquad \xi_i = \phi_i^+ - \phi_i^- \to f \quad \mu\text{-a.e}$

$|\xi_i - f| \longrightarrow 0 \quad \mu\text{-a.e}$

$|\xi_i - f| \le |\xi_i| + |f| \le \widetilde{2|f|} \in L^1(\mu)$

LDC $\qquad \lim_{i \to \infty} \int |\xi_i - f| \, d\mu = 0$

$\Rightarrow \forall \varepsilon > 0 \ \exists i_0 \qquad \int |\xi_{i_0} - f| \, d\mu < \varepsilon$

Alternative proof (H. Chau). Recall for $f \in \mathcal{L}^+$

$$\int f\,d\mu = \sup\left\{ \int \phi\,d\mu : \quad 0 \le \phi \le f \ \text{ & } \phi \text{ simple} \right\}$$

note that if $0 \le \phi \le f \Rightarrow 0 \le \int \phi\,d\mu \le \int f\,d\mu < \infty$

$\forall \varepsilon > 0 \quad \exists \phi_\varepsilon$ simple s.t $0 \le \phi_\varepsilon \le f$

$$0 \le \underbrace{\int f\,d\mu - \int \phi_\varepsilon\,d\mu}_{} < \varepsilon$$

$$= \int (f - \phi_\varepsilon)\,d\mu = \int |f - \phi_\varepsilon|\,d\mu < \varepsilon$$

if $f \in L^1(\mu)$ $\quad f = f^+ - f^- \quad \forall \varepsilon > 0 \quad \phi_\varepsilon^{\pm}$ simple $0 \le \phi_\varepsilon^{\pm} \le f^{\pm}$

$$\int |f^{\pm} - \phi_\varepsilon^{\pm}|\,d\mu < \varepsilon \qquad \phi_\varepsilon = \phi_\varepsilon^+ - \phi_\varepsilon^-$$

$$\int |f - \phi_\varepsilon| = \int |(f^+ - f^-) - (\phi_\varepsilon^+ - \phi_\varepsilon^-)|\,d\mu$$

$$\le \int |f^+ - \phi_\varepsilon^+|\,d\mu + \int |f^- - \phi_\varepsilon^-|\,d\mu < 2\varepsilon \qquad \blacksquare$$