Lebesgue Dominated Convergence Theorem : $(X, \mathcal{M}, \mu)$ complete measure space.

Let $\{f_n\} \subset L^1(\mu)$ be such that

i) $f_n \xrightarrow{n} f$ μ-a.e

ii) there is $g: X \to [0,\infty]$ $g \in L^1(\mu)$ s.t. $|f_n| \le g$ $\forall n$ μ-a.e

Then $f \in L^1(\mu)$ and $\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu$ - Moreover

$\int |f_n - f|\,d\mu \to 0$ as $n \to \infty$ (i.e. $f_n \xrightarrow[n\to\infty]{} f$ in $L^1(\mu)$).

Pf: $f_n \to f$ μ-a.e meas $|f_n| \le g$ $n \to \infty$ $|f| \le g$

$\Rightarrow \int |f|\,d\mu \le \int g\,d\mu < \infty$ $f \in L^1(\mu)$

Fatou : $|f_n| \le g$ $-g \le f_n \le g$

$f_n + g \ge 0$, $g - f_n \ge 0$

$$\int \liminf_{n\to\infty} (f_n + g)\,d\mu \le \liminf_{n\to\infty} \int (f_n + g)\,d\mu$$

$$\int \left(\liminf_{n\to\infty} f_n + g\right) d\mu = \int \left(\lim_{n\to\infty} f_n + g\right) d\mu = \int (f+g)\,d\mu$$

$$\int (f+g)\,d\mu \le \liminf_{n\to\infty} \int (f_n+g)\,d\mu = \liminf_{n\to\infty} \left(\int f_n\,d\mu + \int g\,d\mu\right)$$

$$\boxed{\int f\,d\mu + \cancel{\int g\,d\mu} \le \liminf_{n\to\infty} \int f_n\,d\mu + \cancel{\int g\,d\mu}}$$

$\int g\,d\mu < \infty$

$$\int \liminf_{n\to\infty} (g-f_n)\,d\mu \le \liminf_{n\to\infty} \left(\int g - f_n\right) d\mu$$

$$\int (g-f)\,d\mu = \cancel{\int g\,d\mu} - \int f\,d\mu \le \cancel{\int g\,d\mu} + \liminf_{n\to\infty} -\int f_n\,d\mu$$

$$-\liminf_{n\to\infty} -\int f_n\,d\mu = \boxed{\limsup_{n\to\infty} \int f_n\,d\mu \le \int f\,d\mu}$$

$$g - \limsup_{n\to\infty} f_n = \liminf_{n\to\infty} (g-f_n) \qquad \mu\text{-a.e.}$$

$$\| \qquad \left(\underline{\lim}\,-a_n = -\overline{\lim}\,a_n\right)$$

$$g-f = \liminf_{n\to\infty} (g-f_n)$$

$$\limsup_{n\to\infty} \int f_n \, d\mu \leq \int f \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu$$

$$\leq \limsup_{n\to\infty} \int f_n \, d\mu$$

$$\Rightarrow \quad \limsup_{n\to\infty} \int f_n \, d\mu = \liminf_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$$

$$= \lim_{n\to\infty} \int f_n \, d\mu$$

$$g_n = |f - f_n| \xrightarrow[n\to\infty]{} 0 \qquad \mu\text{-a.e}$$

$$g_n \leq |f| + |f_n| \leq 2g$$

$$\lim_{n\to\infty} \int g_n \, d\mu = \lim_{n\to\infty} \int |f_n - f| \, d\mu = 0$$

Theorem: Suppose $\{f_j\}_j \subset L^1(\mu)$ and $\sum_{j\geq 1} \int |f_j| d\mu < \infty$

then $\sum_{j=1}^{\infty} f_j$ converges μ-a.e and in $L^1(\mu)$ to a function in $L^1(\mu)$

(i.e $S_n = \sum_{j=1}^{n} f_j \in L^1(\mu)$ & $S_n \xrightarrow[n\to\infty]{} S = \sum_{j=1}^{\infty} f_j$ μ-a.e

and $\int |S_n - S| d\mu \xrightarrow[n\to\infty]{} 0$)

Moreover

$$\int \left(\sum_{j=1}^{\infty} f_j\right) d\mu = \sum_{j=1}^{\infty} \int f_j d\mu$$

Pf: MCT $\quad \sigma_n = \sum_{j=1}^{n} |f_j| \leq \sigma_{n+1} = \sum_{j=1}^{n+1} |f_j| \leq \underbrace{\sum_{j=1}^{\infty} |f_j|}_{g}$

$$\lim_{n\to\infty} \int \sigma_n d\mu = \int \lim_{n\to\infty} \sigma_n \, d\mu$$

$$\sum_{j=1}^{\infty} \int |f_j| d\mu = \int \left(\sum_{j=1}^{\infty} |f_j|\right) d\mu \overset{\text{by hyp.}}{<} \infty$$

$$g = \sum_{j=1}^{\infty} |f_j| \quad \in L^1(\mu)$$

$$|S_n| = \left|\sum_{j=1}^{n} f_j\right| \le \sum_{j=1}^{n} |f_j| \le g \quad / g \in L^1(\mu) \quad \mu\text{-a.e } g(x) < \infty$$

$$\sum_{j=1}^{\infty} |f_j(x)| \le g(x) < \infty \Rightarrow \sum_{j=1}^{\infty} f_j(x) \text{ is abs. convergent}$$

$$\sum_{j=1}^{n} f_j(x) \text{ converges} \qquad S_n \longrightarrow S \quad \mu\text{-a.e.}$$

LDCT

$$\int \left(\sum_{j=1}^{\infty} f_j\right) d\mu = \sum_{j=1}^{\infty} \int f_j \, d\mu$$

Theorem (Riesz - Fischer)

$L^1(\mu)$ is a complete metric space.

Definitions: ① Given a vector space X, a norm on X is a function

$$\| \ \| : X \longrightarrow [0, \infty) \quad \text{satisfying}$$

i) $\|x\| = 0 \iff x = 0$

ii) $\|\lambda x\| = |\lambda| \|x\| \qquad \forall x \in X \qquad \forall \lambda \in \mathbb{R} \ (\mathbb{C})$

iii) $\|x + y\| \leq \|x\| + \|y\|$ (Triangle inequality)

② A norm produces a metric on X $\quad \rho(x,y) = \|x - y\|$

③ If $\{x_n\}_{n=1}^{\infty} \subset X$

i) $\sum_{n=1}^{\infty} x_n$ converges to x if $\sum_{n=1}^{N} x_n \longrightarrow x$ as $N \to \infty$

ii) $\sum_{n \geq 1} x_n$ is absolutely convergent if $\sum_{n \geq 1} \|x_n\| < \infty$

Theorem: A normed vector space is complete iff every absolutely convergent series in X converges in X.

Theorem (Riesz - Fischer)

$L^1(\mu)$ is a complete metric space.

Theorem ① Suppose $\{f_j\}_j \subset L^1(\mu)$ and $\sum_{j\geq 1} \int |f_j| d\mu < \infty$

then $\sum_{j=1}^{\infty} f_j$ converges μ-a.e and in $L^1(\mu)$ to a function in $L^1(\mu)$

Theorem ② A normed vector space is complete iff every absolutely convergent series in X converges in X.

Proof of Riesz-Fischer: $\{f_j\} \subset L^1(\mu)$ $\sum_{n=1}^{\infty} \int |f_j| d\mu < \infty$ ($\int |f_j| d\mu = \|f_j\|_{L^1}$)

by ① $\Rightarrow$ $\sum_{j=1}^{\infty} f_j$ converges in $L^1(\mu)$ $\Rightarrow$ ② $L^1(\mu)$ is complete.

Theorem: A normed vector space is complete iff every absolutely convergent series in X converges in X.

Pf: $\Rightarrow$ | X complete & $\sum_{n=1}^{\infty} \|x_n\| < \infty$

$$S_N = \sum_{n=1}^{N} x_n$$

$$\|S_N - S_{N+k}\| = \left\| \sum_{j=N+1}^{N+k} x_j \right\| \leq \sum_{j=N+1}^{N+k} \|x_j\|$$

tail of ↑ convergent series can be as small as we want

$\{S_N\}$ is Cauchy

X is complete

$$S_N \longrightarrow S = \sum_{n=1}^{\infty} x_n \in X.$$

Corollary: Let $\{f_n\} \subset L^1(\mu)$ & assume that $f_n \to f$ in $L^1(\mu)$ then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ μ a.e.

Theorem: If $f \in L^1(\mu)$; given $\varepsilon > 0$ there exists ϕ integrable and simple such that

$$\int |\phi - f| \, d\mu < \varepsilon$$

i.e. simple integrable functions are dense in $L^1(\mu)$.