HOMEWORK 10

TI problems due Wednesday December 9 at 10pm PT

Problems from Folland: Chapter 2 problem 64.

Problem 1 (Prelim 2016) Suppose that a (x, d) is a compact metric space and $f : X \to \mathbb{R}$ is a (not necessarily continuous) function. If for every $t \in \mathbb{R}$ $f^{-1}([t, \infty))$ is closed, then there is some $x_0 \in X$ so that $f(x_0) = \sup_{x \in X} f(x) < \infty$.

Problem 2: Prove that if a measurable subset $E \subset [0, 1]$ satisfies $m(E \cap I) \geq \alpha m(I)$ for some $\alpha > 0$ and all intervals I in [0,1], then E has measure 1.

Definition: A measure μ in $\mathbb{R}^n$ us doubling if there exists a constant $C \geq 1$ such that for every $x \in \mathbb{R}^n$ and every $r > 0$

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

Note: μ is not necessarily Lebesgue measure.

Problem 3: Let μ be a doubling measure on $\mathbb{R}^n$. Assume that $\mu(B(x_0, 1)) < \infty$ for some $x_0 \in X$.

1. Show that if K is compact then $\mu(K) < \infty$.
2. Prove that if $E \in \mathcal{B}_{\mathbb{R}^n}$ with $\mu(E) < \infty$ then given $\epsilon > 0$ there exists a compact set K with $K \subset E$ and $\mu(E \backslash K) < \epsilon$.

Problem 4: Show that for any open set $U \subset \mathbb{R}^n$, and $\delta > 0$, there exists a countable collection $\mathcal{G}$ of disjoint closed balls in U such that $\operatorname{diam} \mathrm{B} \leq \delta$ for all $B \in \mathcal{G}$, and

$$m(U \backslash \bigcup_{B \in \mathcal{G}} B) = 0.$$