Problem 1: Integrability of f in $\mathbb{R}$ does not necessarily imply the convergence of $f(x)$ to 0 as $x \to \infty$.

1. There exists a continuous non-negative function f on $\mathbb{R}$ so that f is integrable on $\mathbb{R}$ but $\limsup_{x \to \infty} f(x) = \infty$.

2. However if we assume that f is uniformly continuous on $\mathbb{R}$ and integrable, then $\limsup_{|x| \to \infty} f(x) = 0$.

1)

$$f_n(x) = \begin{cases} n & x \in \left[n, n+\frac{1}{n^3}\right] \quad n \geq 2 \\ n^4\left(x - n + \frac{1}{n^3}\right) & x \in \left[n-\frac{1}{n^3}, n\right) \\ n^4\left(n + \frac{2}{n^3} - x\right) & x \in \left(n+\frac{1}{n^3}, n+\frac{2}{n^3}\right) \\ 0 & x \notin \left(-\infty; n-\frac{1}{n^3}\right) \cup \left(n+\frac{2}{n^3}, \infty\right) \end{cases}$$

$$f(x) = \sum_{n\geq 2}^{\infty} f_n(x) \qquad \sum_{n\geq 2}^{k_2} f_n(x) \leq \sum_{n\geq 2}^{k+1} f_n(x)$$

$$\int_{-\infty}^{\infty} f(x)dx = \sum_{n\geq 2}^{\infty} \int_{-\infty}^{\infty} f_n(x)dx \qquad \downarrow \text{MCT}$$

$$= \sum_{n\geq 2}^{\infty} 2\frac{n}{n^3} = \sum_{n\geq 2}^{\infty} 2/n^2 < \infty$$

$$\inf_{n\geq 1} \sup_{|x|\geq n} f(x) = +\infty$$

2) f unif. continuous $\quad \forall \varepsilon > 0 \quad \exists \delta > 0 \quad |x - x_0| < \delta$

$$\Rightarrow |f(x) - f(x_0)| < \varepsilon$$

assume $\limsup_{|x| \to \infty} f(x) > \eta > 0$ wlog $x \to \infty$

$\Downarrow$ $\forall n$

$\exists x_n \to +\infty \quad f(x_n) \geq \eta \quad \varepsilon = \eta/2 \quad \exists \delta \in (0, 1/2)$

wlog $x_n \nearrow$ & $|x_n - x_{n+1}| \geq 1$

$$|x - x_n| < \delta \Rightarrow |f(x) - f(x_n)| < \eta$$

$$\Rightarrow \forall x \in (x_n - \delta, x_n + \delta) \quad |f(x)| \geq \eta/2$$

$$(x_n - \delta, x_n + \delta) \cap (x_\ell - \delta, x_\ell + \delta) = \emptyset$$

$$\int_{\mathbb{R}} |f(x)|\,dx \geq \sum_{n=1}^{\infty} \int_{x_n - \delta}^{x_n + \delta} |f(x)|\,dx \geq \sum_{n=1}^{\infty} \frac{\eta}{2} \cdot 2\delta = +\infty$$

$*$

Proposition: Let $(X, \mathcal{M}, \mu)$ be a complete measure space. If $f \in L^1(\mu)$ then

① $\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t. $\int_E |f| d\mu < \varepsilon$ whenever $\mu E < \delta$, $E \in \mathcal{M}$

② if μ is σ-finite $\forall \varepsilon > 0 \quad \exists B \in \mathcal{M}$ so that

$\mu B < \infty$ & $\int_{B^c} |f| d\mu < \varepsilon$

Pf: ① (if f where bounded $|f| \leq M$

$\int_E |f| d\mu \leq \int_E M \, d\mu \leq M \mu E < \varepsilon$ if $\mu E < \varepsilon / M = \delta$)

w.log. $f \geq 0$ ($g = |f|$) for $n \in \mathbb{N}$

$E_n = \{x : f(x) \geq n\}$ & $\mu\{x : f(x) = \infty\} = 0$ (*)

$\int f(x) d\mu \geq \int_{E_n} f(x) d\mu \geq n \mu E_n \Rightarrow \mu E_n \leq \frac{1}{n} \int f(x) d\mu$ (Chebyshev)

$\mu E_n \to 0$ as $n \to \infty$

$E_{n+1} \subset E_n \qquad E_n \setminus E_{n+1}$

$$\infty > \int f(x)\,d\mu = \sum_{n=0}^{\infty} \int_{E_n \setminus E_{n+1}} f(x)\,d\mu$$

$F_n = \{x : f(x) < n\} \qquad F_n \subset F_{n+1} \qquad f_n = f\chi_{F_n}$

$0 \le f_n \le f_{n+1} \le \cdots \le f \qquad f_n \to f \quad \mu\text{-a.e}$ by (x)

MCT $\qquad \int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu$

Given $\varepsilon > 0 \qquad \exists N \in \mathbb{N} \qquad \forall n \ge N$

$\int f\,d\mu - \int f_n\,d\mu < \varepsilon/2$ in particular $\int f\,d\mu - \int f_N\,d\mu < \varepsilon/2$

$$\int_E f\,d\mu = \int_E \underbrace{(f - f_N)}_{\ge 0}\,d\mu + \int_E f_N\,d\mu < \frac{\varepsilon}{2} + N\mu E < \varepsilon$$

whenever $\mu E < \frac{\varepsilon}{2N} = \delta$

(2) if μ is σ-finite $\forall \varepsilon > 0$ $\exists B \in \mathcal{M}$ so that

$\mu B < \infty$ & $\int_{B^c} |f| d\mu < \varepsilon$ w log $f \geq 0$

$X = \bigcup_j \tilde{B}_j$ $\quad B_n = \bigcup_{j=1}^{n} \tilde{B}_j$ $\quad B_n \subset B_{n+1}$

$\mu(\tilde{B}_j) < \infty \Rightarrow \mu(B_n) < \infty$

$f_n = f \chi_{B_n}$ $\quad 0 \leq f_n \leq f_{n+1} \leq \ldots \leq f$

$f_n \longrightarrow f$

MCT

$$\lim_{n \to \infty} \int f_n d\mu = \int f d\mu$$

$\forall \varepsilon > 0$ $\exists N$ $\int f d\mu - \int f_N d\mu < \varepsilon$

$$= \int f d\mu - \int_{B_N} f d\mu = \int_{B_N^c} f d\mu < \varepsilon$$

Lebesgue Dominated Convergence Theorem : $(X, \mathcal{M}, \mu)$ complete measure space.

Let $\{f_n\} \subset L^1(\mu)$ be such that

i) $f_n \xrightarrow{n} f$ μ-a.e

ii) there is $g: X \to [0,\infty]$ $g \in L^1(\mu)$ s.t. $|f_n| \leq g$ $\forall n$ μ-a.e

Then $f \in L^1(\mu)$ and $\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$ - Moreover

$\int |f_n - f| d\mu \to 0$ as $n \to \infty$ (i.e. $f_n \xrightarrow[n\to\infty]{} f$ in $L^1(\mu)$).

Theorem: Suppose $\{f_j\}_j \subset L^1(\mu)$ and $\sum_{j\geq 1} \int |f_j| d\mu < \infty$

then $\sum_{j=1}^{\infty} f_j$ converges μ-a.e and in $L^1(\mu)$ to a function in $L^1(\mu)$

(ie $S_n = \sum_{j=1}^{n} f_j \in L^1(\mu)$ & $S_n \xrightarrow[n\to\infty]{} S = \sum_{j=1}^{\infty} f_j$ μ-a.e

and $\int |S_n - S| d\mu \xrightarrow[n\to\infty]{} 0$)

Moreover

$$\int \left(\sum_{j=1}^{\infty} f_j\right) d\mu = \sum_{j=1}^{\infty} \int f_j d\mu$$

Problem 2: (Prelim 2000) Show in detail that

$$\int_0^1 \sum_{k=1}^{\infty} \frac{x^k \cos(2^k \pi x)}{k}\, dx = \sum_{k=1}^{\infty} \int_0^1 \frac{x^k \cos(2^k \pi x)}{k}\, dx.$$

$\in L^1([0,1])$