Fatou's lemma: If $\{f_n\} \subset \mathcal{L}^+$ then

$$\int \left(\liminf_{n\to\infty} f_n\right) d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu$$

Pf: $\inf_{k\geq n} f_k = g_n$ $\quad 0 \leq g_n \leq g_{n+1} \leq f$ measurable

$$g_n \xrightarrow[n\to\infty]{} \liminf_{k\to\infty} f_k = f$$

MCT

$$\int f \, d\mu = \lim_{n\to\infty} \int g_n \, d\mu$$

$$= \sup_{n\geq 1} \int g_n \, d\mu$$

$$= \sup_{n\geq 1} \inf_{k\geq n} f_k$$

$$= \liminf_{k\to\infty} f_k$$

$0 \leq g_n \leq f_k \quad k \geq n$

$$\int g_n \, d\mu \leq \int f_n \, d\mu$$

$$\int g_n \, d\mu \leq \inf_{k\geq n} \int f_n \, d\mu$$

Corollary: If $\{f_n\} \subset \mathcal{L}^+$, $f \in \mathcal{L}^+$ & $f_n \to f$ μ-a.e

$$\int f \, d\mu \leq \liminf_{n \to \infty} \int f_n \, d\mu$$

Pf: $f = \liminf_{n \to \infty} f_n$ μ-a.e

(trick from pointwise to a.e convergence + Fatou)

Proposition (HW): If $f \in \mathcal{L}^+$ & $\int f \, d\mu < \infty$ then

$\{x : f(x) = +\infty\}$ is a set of μ-measure 0

and $\{x : f(x) > 0\}$ is σ-finite.

Integration of extended real valued functions

$(X, \mathcal{A}, \mu)$ complete measure space - Let $f: X \to \overline{\mathbb{R}}$ measurable if $f^{\pm} = \max\{0; \pm f\} \in \mathcal{L}^+$

If either $\int f^+ d\mu < \infty$ or $\int f^- d\mu < \infty$ we define

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

Definition: Let $f: X \to \overline{\mathbb{R}}$ measurable, we say that f is integrable w.r.t μ if

$$\int |f| d\mu = \int f^+ d\mu + \int f^- d\mu < \infty$$

Proposition: If f, g are measurable, $\int |f| d\mu < \infty$ and $\int |g| d\mu < \infty$ then for $\alpha, \beta \in \overline{\mathbb{R}}$

$$\int (\alpha f + \beta g) d\mu = \alpha \int f d\mu + \beta \int g d\mu$$

Proposition: If f is measurable

$$\left| \int f d\mu \right| \leq \int |f| d\mu$$

Pf

$$\left| \int f d\mu \right| = \left| \int \underbrace{f^+ d\mu} - \int \underbrace{f^- d\mu} \right|$$

$$\leq \left| \int f^+ d\mu \right| + \left| \int f^- d\mu \right|$$

$$\leq \int f^+ d\mu + \int f^- d\mu$$

$$\leq \int (f^+ + f^-) d\mu = \int |f| d\mu$$

Proposition (H+W) If $\int |f| d\mu < \infty$ then

$\{ |f| = +\infty \}$ has μ-measure 0 &

$\{ |f| \neq +\infty \}$ is σ-finite

Proposition: If f, g are measurable, $\int |f| d\mu < \infty$ & $\int |g| d\mu < \infty$ then the following are equivalent

$\rightsquigarrow$ i) $\int_E f d\mu = \int_E g d\mu \quad \forall E \in \mathcal{M}$

ii) $\int |f - g| d\mu = 0$

iii) $f = g \quad \mu$ a.e.

$||f| - |g|| \leq |f - g|$

$\left| \int |f| d\mu - \int |g| d\mu \right| \leq \int |f - g| d\mu$

Pf (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (i) (ii) $\Rightarrow$ (iii) $\int \underbrace{|f - g|}_{|f-g| \geq 0} d\mu = 0 \Leftrightarrow |f - g| = 0$ μ a.e

$\Leftrightarrow f = g \quad \mu$ a.e

iii) $\Rightarrow$ i) $f = g \quad \mu$ a.e $\quad f\chi_E = g\chi_E \quad \mu$ a.e

ii) $\Rightarrow \int |\chi_E f - \chi_E g| d\mu = 0 \Rightarrow \left| \int (\chi_E f - \chi_E g) d\mu \right|$

previous prop $\leq \rightarrow \leq 0$

$\int \chi_E f d\mu = \int_1 \chi_E g d\mu$

ii) $\Rightarrow$ iii) $\quad \int_E f\,d\mu = \int_E g\,d\mu \quad \forall E \in \mathcal{M} \overset{?}{\Rightarrow} f = g$ μ-a.e

assume $\quad \{x : f(x) \neq g(x)\} = E = \underbrace{\{x : (f-g)(x) > 0\}}_{E_1}$

E measurable $\Leftarrow$ E_1 measurable $\cup$ $\underbrace{\{x : (f-g)(x) < 0\}}_{E_2}$ measurable

$\mu E > 0$

either $\mu E_1 > 0$ or $\mu E_2 > 0$

$(f-g)\chi_{E_1} \geq 0 \Rightarrow \int_{E_1} (f-g)\,d\mu > 0$ (is does hold)

(if $\int (f-g)\chi_{E_1}\,d\mu = 0 \Rightarrow (f-g)\chi_{E_1} = 0$ μ a.e

but $f-g > 0$ on E_1

and $\mu E_1 > 0$)

If $f, g : X \to \overline{\mathbb{R}}$ measurable define $f \sim g$ if $f = g$ μ-a.e

$$[f] = \{ g : X \to \overline{\mathbb{R}} : f = g \ \mu\text{-a.e} \}$$

$$L^1(\mu) = \{ [f] : \int |f| \, d\mu < \infty \}$$

if $g \in [f]$ $\quad f = g$ μ-a.e $\Rightarrow \int |f| \, d\mu = \int |g| \, d\mu$

we drop [ ]

Define $\rho : L^1(\mu) \times L^1(\mu) \to \mathbb{R}^+$

$$\rho(f, g) = \int |f - g| \, d\mu$$

$\rho(f, g) = 0 \iff \int |f - g| \, d\mu = 0 \Rightarrow f = g$ (μ-a.e)

$(L^1(\mu), \rho)$ metric space

Note: $f_n \xrightarrow[n\to\infty]{} f$ in $L^1(\mu)$ iff $\int |f_n - f| \, d\mu \xrightarrow[n\to\infty]{} 0$

We have 4 modes of convergence:

i) uniform
ii) pointwise
iii) a.e
iv) $L^1(\mu)$

how do they relate?

($\mathbb{R}$, $\mathcal{L}$, m)

a) $f_n = \frac{1}{n} \chi_{(0,n)}$ b) $f_n = \chi_{(n,n+1)}$ c) $f_n = n \chi_{(0,\frac{1}{n})}$

d) $f_1 = \chi_{[0,1]}$ $f_2 = \chi_{[0,1/2]}$ $f_3 = \chi_{[1/2,1]}$

$f_4 = \chi_{[0,1/4]}$ $f_5 = \chi_{[1/4,1/2]}$ $f_6 = \chi_{[1/2,3/4]}$ $f_7 = \chi_{[\frac{3}{4},1]}$

$f_n = \chi_{\left[\frac{j}{2^k}, \frac{j+1}{2^k}\right]}$ where $n = 2^k + j$ for $0 \le j < 2^k$.

Proposition: Let $(X, \mathcal{M}, \mu)$ be a complete measure space. If $f \in L^1(\mu)$ then

(1) $\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t. $\int_E |f| \, d\mu < \varepsilon$ whenever $\mu E < \delta$, $E \in \mathcal{M}$

(2) If μ is σ-finite $\forall \varepsilon > 0 \quad \exists B \in \mathcal{M}$ so that

$$\int_{B^c} |f| \, d\mu < \varepsilon$$

Problem 1: Integrability of f in $\mathbb{R}$ does not necessarily imply the convergence of $f(x)$ to 0 as $x \to \infty$.

1. There exists a continuous non-negative function f on $\mathbb{R}$ so that f is integrable on $\mathbb{R}$ but $\limsup_{x \to \infty} f(x) = \infty$.

2. However if we assume that f is uniformly continuous on $\mathbb{R}$ and integrable, then $\limsup_{|x| \to \infty} f(x) = 0$.