Monotone convergence theorem (MCT)

If $\{f_n\}$ is a sequence in $\mathcal{L}^+$ such that

$$0 \le f_n \le f_{n+1} \quad \forall$$

$$f = \lim_{n \to \infty} f_n = \sup_{n \ge 1} f_n \quad \text{then}$$

$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu$$

Pf: to show $\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu$ we need to show 2 inequalities; recall $f_n \in \mathcal{L}^+$ and $0 \le f_n \le f_{n+1} \le \cdots \le f$

$\forall \; f = \sup_{n \ge 1} f_n$ -

Pf: for $x \in X$ $\{f_n(x)\}_n$ increasing sequence thus

$f(x) = \lim_{n \to \infty} f_n(x)$ exists possibly $+\infty$ but since each f_n measurable, f measurable.

$$0 \leq f_n \leq f_{n+1} \leq \cdots \leq f$$

$$\Rightarrow \quad 0 \leq \int f_n \, d\mu \leq \int f_{n+1} \, d\mu \leq \cdots \leq \int f \, d\mu$$

$\left\{\int f_n \, d\mu\right\}_n$ non-negative increasing sequence thus

$$\lim_{n \to \infty} \int f_n \, d\mu \leq \int f \, d\mu$$

exists (possibly $+\infty$)

Reverse inequality : let $\alpha \in (0,1)$ ϕ simple s.t $0 \le \phi \le f$

$E_n = \{ x \in X : f_n(x) \ge \alpha \phi(x) \}$ measurable

$E_n = \{ (f_n - \alpha\phi)^{-1} \ge 0 \}$

since $f_{n+1} \ge f_n$ what can you say about the E_n's ?

$E_n \subset E_{n+1}$

since $f_n \to f$ and $f \ge \phi \ge \alpha\phi$ what else can we say about the E_n's ?

$\bigcup_n E_n = X$ $\quad$ ($x \in X$ $\quad f_n(x) \to f(x)$

either $f(x) = 0$ $\quad f_n(x) = 0$

$\phi(x) \Rightarrow x \in E_n$

or $f(x) > 0$ $\quad f(x) > \alpha\phi(x)$

for n large enough

$f_n(x) > \alpha\phi(x)$

Reverse inequality : let $\alpha \in (0,1)$ ϕ simple s.t $0 \le \phi \le f$

$$E_n = \{ x \in X : f_n(x) \ge \alpha \phi(x) \} \quad \text{measurable}$$

since $f_{n+1} \ge f_n$; $E_n \subset E_{n+1}$ ↑

since $f_n \to f$ ↑ $f \ge \phi \ge \alpha\phi$ then $X = \bigcup_n E_n$

$$\int f_n \, d\mu = \int f_n \, d\mu = \int_{E_n} f_n d\mu + \int_{E_n^c} f_n d\mu$$

$E_n \cup E_n^c$ ← (*)

because

$$\int_E f_n \, d\mu = \nu_n(E) \ge \int f_n \chi_{E_n} d\mu = \int_{E_n} f_n d\mu \ge \alpha \int_{E_n} \phi \, d\mu$$

ν_n is a measure

$$\int f_n d\mu \ge \alpha \int_{E_n} \phi \, d\mu = \alpha \nu(E_n)$$

Recall $\nu(E) = \int_E \phi \, d\mu$ is a measure then since

$E_n \subset E_{n+1}$ $\quad \nu(X) = \nu\left(\bigcup_n E_n\right) = \lim_{n \to \infty} \nu(E_n)$

$$\nu(X) = \liminf_{n\to\infty} \nu(E_n) \quad \& \quad \int f_n \, d\mu \geq \alpha \int_{E_n} \phi \, d\mu = \alpha \, \nu(E_n)$$

$$\liminf_{n\to\infty} \int f_n \, d\mu \geq \alpha \, \nu(X) = \alpha \int_X \phi \, d\mu \qquad \forall \alpha \in (0,1)$$

letting $\alpha \to 1$

$$\lim_{n\to\infty} \int f_n \, d\mu \geq \int_X \phi \, d\mu \qquad \text{for every } \phi \text{ simple } 0 \leq \phi \leq f$$

$$\lim_{n\to\infty} \int f_n \, d\mu \geq \underbrace{\sup_{\phi} \left\{ \int \phi \, d\mu : 0 \leq \phi \leq f, \ \phi \text{ simple} \right\}}_{\int f \, d\mu} \qquad \blacksquare$$

Q: Why is this theorem useful? Thm 2.10 $f \in \mathcal{L}^+$

$0 \leq \phi_1 \leq \dots \leq \phi_n \leq \phi_{n+1} \leq f$ // simple

$\phi_n \to f$

$$\Rightarrow \int f \, d\mu = \lim_{n\to\infty} \int \phi_n \, d\mu$$

Theorem: If $\{f_n\}$ is a finite or infinite sequence in $\mathcal{L}^+$ & $f = \sum_n f_n$ then $\int f\,d\mu = \sum_n \int f_n\,d\mu$

In particular $$\int (f_1 + f_2)\,d\mu = \int f_1\,d\mu + \int f_2\,d\mu \quad (*)$$

Pf: $\psi_n = \sum_{k=1}^{n} f_k \qquad 0 \le \psi_n \le \psi_{n+1} \le \dots \le f$

$\psi_n \to f$

apply the theorem

$$\lim_{n\to\infty} \int \psi_n\,d\mu = \lim_{n\to\infty} \int \left(\sum_{k=1}^{n} f_k\,d\mu\right) =$$

I want to exchange $\int$ & $\sum_{k=1}^{n}$, what do I need?

we need to show $(*)$ then we apply induction for finitely many

$0 \le \phi_1 \le \dots \le \phi_n \le \phi_{n+1} \le \dots \le f_1 \qquad f_1 = \lim \phi_n$

(ϕ_1: simple; ϕ_n: simple)

$$0 \le \psi_1 \le \dots \le \psi_n \le \psi_{n+1} \le \dots \le f_2 \qquad f_2 = \lim \psi_n$$

$\uparrow$ simple

$$0 \le \phi_1 + \psi_1 \le \dots \le \phi_n + \psi_n \le \phi_{n+1} + \psi_{n+1} \le \dots \le f_1 + f_2$$

$\uparrow$ simple

$$f_1 + f_2 = \lim_{n \to \infty} \phi_n + \psi_n$$

$$\int (f_1 + f_2)\, d\mu \overset{\text{MCT}}{=} \lim_{n \to \infty} \int (\phi_n + \psi_n)\, d\mu \overset{\text{linearity, simple functions}}{=}$$

$$= \lim_{n \to \infty} \int \phi_n \, d\mu + \int \psi_n \, d\mu$$

MCT $\downarrow$

$$= \int f_1 \, d\mu + \int f_2 \, d\mu \qquad \sum_{k=1}^{n} \int f_k \, d\mu$$

$$\lim_{n \to \infty} \int \psi_n \, d\mu = \lim_{n \to \infty} \int \left(\sum_{k=1}^{n} f_k \, d\mu \right) = \lim_{n \to \infty} \sum_{k=1}^{n} \int f_k \, d\mu$$

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Proposition: If $f \in \mathcal{L}^+$ then $\int f d\mu = 0$ iff $f = 0$ μ-a.e

(i.e $\mu\{x : f(x) \neq 0\} = 0$)

Pf $\Rightarrow$ $\int f d\mu = 0$ then $E = \{x : f(x) > 0\}$ ~;

$$\int f d\mu = \int_E f d\mu + \underbrace{\int_{E^c} f d\mu}_{0} = \int_E f d\mu \geq \int_{E_n} f d\mu$$

$$\geq \frac{1}{n} \mu E_n$$

$$E_n = \{x : f(x) > \tfrac{1}{n}\} \subset E$$

$$0 = \int f d\mu \geq \frac{1}{n} \mu E_n \Rightarrow \mu E_n = 0 \quad \forall$$

$$\bigcup_{n=1}^{\infty} E_n = E$$

$$\mu E \leq \sum_n \mu E_n = 0$$

$\Leftarrow$ $f = 0$ μ a.e $\quad 0 \leq \phi \leq f \Rightarrow \phi = 0$ μ-a.e. (ϕ: simple function)

$$\phi = \sum_{i=1}^{n} \alpha_i \chi_{F_i} \quad \text{if } \mu F_i > 0 \Rightarrow \alpha_i = 0 \quad \int \phi d\mu = \sum_{i=1}^{n} \alpha_i \mu F_i = 0$$

Corollary: Let $\{f_n\} \subset \mathcal{L}^+$, $f \in \mathcal{L}^+$ and

$f_n(x) \le f_{n+1}(x)$ μ-a.e x & $f(x) = \lim_{n\to\infty} f_n(x)$ μ-a.e x

then $$\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu \qquad \text{(MCT)}$$

Q: What is the difference with the MCT?

$E = \{x : f_n(x) \nearrow f(x)\}$ $\quad \mu E^c = 0$

$f_n \chi_E \le f_{n+1} \chi_E \le f \chi_E$ $\quad \{f_n \chi_E\}$ MCT

$f - f\chi_E = 0$ μ-a.e $\quad \int f\,d\mu = \int f \chi_E\,d\mu$

$f_n - f_n \chi_E = 0$ μ-a.e $\quad \int f_n\,d\mu = \int f_n \chi_E\,d\mu$

$$\int f\,d\mu = \int f\chi_E\,d\mu = \lim_{n\to\infty} \int f_n \chi_E\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu.$$