Theorem (2.10) Let $(X, \mathcal{M})$ be a measurable space

a) If $f : X \longrightarrow [0, \infty]$ is measurable there is a sequence $\{\phi_n\}_n$ of simple functions s.t.

$$0 \leq \phi_1 \leq \phi_2 \leq \ldots \leq \phi_n \leq \ldots \leq f$$

with

$$\phi_n \xrightarrow[n \to \infty]{} f \quad \text{pointwise} \qquad f \quad \phi_n \xrightarrow[n \to \infty]{} f \quad \text{uniformly on sets on which } f \text{ is bounded}$$

b) If $f : X \longrightarrow \overline{\mathbb{R}}$ is measurable there is a sequence $\{\phi_n\}$ of simple functions s.t

$$0 \leq |\phi_1| \leq |\phi_2| \leq \ldots \leq |\phi_n| \leq \ldots \leq |f|$$

with

$$\phi_n \xrightarrow[n \to \infty]{} f \quad \text{pointwise} \qquad f \quad \phi_n \xrightarrow[n \to \infty]{} f \quad \text{uniformly on sets on which } f \text{ is bounded}$$

Proof : a) For $n = 0, 1, 2$ & $0 \le k \le 2^{2n} - 1$ let

$$E_n^k = f^{-1}\left(k2^{-n} ; (k+1)2^{-n}\right] \qquad F_n = f^{-1}(2^n, \infty)$$

$$\phi_n = \sum_{k=0}^{2^{2n}-1} k2^{-n} \chi_{E_n^k} + 2^n \chi_{F_n}$$

Claim 2 $\quad 0 \le f - \phi_n \le 2^{-n}$ on the set where $f < 2^n$

let $x \in X \quad f(x) < 2^n \quad \exists\, k \in \{0, 1, \dots, 2^{2n} - 1\}$ s.t

$$k2^{-n} < f(x) \le (k+1)2^{-n} \qquad (\text{if } f(x) = 0 \checkmark \phi_n(x) = 0)$$

$$\phi_n(x) = k2^{-n}$$

$$0 \le f(x) - \phi_n(x) \le (k+1)2^{-n} - k2^{-n} = 2^{-n}$$

Given $\varepsilon > 0$
$f(x) < M$ choose n_0 s.t $2^{n_0} > M \quad \frac{1}{2^{n_0}} < \varepsilon$

$\forall n \ge n_0 \quad 0 \le f(x) - \phi_n(x) \le f(x) - \phi_{n_0}(x) \le 2^{-n_0} < \varepsilon$

2) $f: X \to \overline{\mathbb{R}}$ $\quad f = f^+ - f^-$ $\quad f^\pm$ measurable

$0 \le \phi_1^\pm \le \phi_2^\pm \le \dots \le f^\pm$ by previous construction

$|f| = f^+ + f^-$ $\quad \phi_n^\pm \longrightarrow f^\pm$

$0 \le |\phi_1| \le \dots \le |\phi_n| \le |\phi_{n+1}| \le \dots \le |f|$

(uniform part the same)

b) $f : X \to \overline{\mathbb{R}}$ then $f = f^+ - f^-$ $\quad |f| = f^+ + f^-$

by the previous construction there exist simple functions s.t.

$$0 \le \phi_1^{\pm} \le \phi_2^{\pm} \le \dots \le \phi_n^{\pm} \le \dots \le f^{\pm}$$

$\phi_n^{\pm} \xrightarrow[n \to \infty]{} f^{\pm}$ pointwise + uniformly when $f^{\pm}$ is bounded

Moreover $0 \le \phi_1^+ + \phi_1^- \le \dots \le \phi_n^+ + \phi_n^- \le \dots \le f^+ + f^-$ ∎

Proposition: Let $(X, \mathcal{M}, \mu)$ be a measure space. The following implications are valid iff μ is complete.

1. If $f : X \to \overline{\mathbb{R}}$ is measurable & $f = g$ μ-a.e then g is μ-measurable

(μ-a.e means μ-almost everywhere that is $f = g$ except for a set of μ-measure 0 i.e $\mu\{x : f(x) \neq g(x)\} = 0$)

2. If $\{f_n\}$ is a sequence of measurable functions and $f_n \to f$ μ.a.e then f is measurable

Pf : if μ is not complete : $\exists E \subset F$ $\mu(F) = 0$ $F \in \mathcal{M}$

if $E \notin \mathcal{M}$ $\chi_E = 0$ μ a.e because $\mu E \leq \mu F = 0$

↑ is not measurable

Q: How do you "complete" a measure?

all set of measure 0 are measure so add them to the σ-algebra. (you have new σ-algebra that makes the measure complete)

if $F \in \mathcal{M}$ and $\mu F = 0$ add to 2^F to the σ-algebra. $\forall E \subset F$ $\mu E = 0$

Integration of non-negative functions

$(X, \mathcal{M}, \mu)$ complete measure space $\quad \mathcal{M}^+ = \{ f : X \to [0, \infty) \text{ measurable} \}$

Definition: If $\phi = \sum_{i=1}^{m} \alpha_i \chi_{E_i}$ is a simple function with standard representation we define the integral of ϕ w.r.t μ as

$$\int_X \phi \, d\mu = \int \phi \, d\mu = \sum_{i=1}^{m} \alpha_i \mu E_i$$

Remark: If ϕ is simple & $A \in \mathcal{M}$ so is $\phi \chi_A = \sum_{i=1}^{m} \alpha_i \chi_{A \cap E_i}$

and

$$\int_A \phi \, d\mu = \int \phi \chi_A \, d\mu$$

Proposition: Let ϕ & ψ be simple functions in $\mathcal{L}^+$

i) If $c \geq 0$ $\quad \int c\phi \, d\mu = c \int \phi \, d\mu$

ii) $\int (\phi + \psi) \, d\mu = \int \phi \, d\mu + \int \psi \, d\mu$

iii) If $0 \leq \phi \leq \psi$ $\quad \int \phi \, d\mu \leq \int \psi \, d\mu$

iv) If $\nu(A) = \int_A \phi \, d\mu$ for $A \in \mathcal{M}$ then ν is a measure on $\mathcal{M}$.

Breakout rooms: 1) & 2) prove ii)

3) & 4) prove iii)

5) & 6) prove iv)

Definition: If $f \in \mathcal{L}^+$ define

$$\int f \, d\mu = \sup \left\{ \int \phi \, d\mu : 0 \le \phi \le f : \phi \text{ simple function} \right\}$$

Remark: If $\phi \in \mathcal{L}^+$ is simple there are 2 definitions of integral

① $\phi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$ $\qquad \int \phi \, d\mu = \sum_{i=1}^{n} \alpha_i \mu E_i$

② $\int \phi \, d\mu = \sup \left\{ \underbrace{\int \psi \, d\mu}_{①_\psi} : 0 \le \psi \le \phi : \psi \text{ simple function} \right\}$

Why are they the same?

ϕ is simple $\qquad ①_\phi \le ②$

if ψ simple $\qquad \psi \le \phi \implies \int \psi \, d\mu \le \int \phi \, d\mu$

$①_\psi \le ①_\phi$

take $\sup_\psi$ $\qquad ② \le ①_\phi$

Definition: If $f \in \mathcal{L}^+$ define

$$\int f \, d\mu = \sup \left\{ \int \phi \, d\mu : 0 \leq \phi \leq f \; : \phi \text{ simple function} \right\}$$

Properties: i) if $f, g \in \mathcal{L}^+$, $f \leq g$ then

$$\int f \, d\mu \leq \int g \, d\mu$$

ii) if $c \geq 0$ $\quad \int cf \, d\mu = c \int f \, d\mu$

Recall we want Lebesgue measure to be a generalization of Riemann integration

What property are we missing?

Monotone convergence theorem

If $\{f_n\}$ is a sequence in $\mathcal{L}^+$ such that

$$0 \le f_n \le f_{n+1} \quad \forall$$

$$f = \lim_{n \to \infty} f_n = \sup_{n \ge 1} f_n \quad \text{then}$$

$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu$$

Hypotheses are crucial : 1) $f_n \ge 0$ if not $f_n = -\chi_{(-\infty, -n)}$

$$f_n \le f_{n+1} \le \dots \le 0 \qquad f_n \to 0$$

$$\int f_n \, dx = -\infty \neq \int 0 \, dx$$

2) $\{f_n\}$ increasing

a) $f_n = \chi_{[n,n+1]}$ $\qquad f_n \longrightarrow 0$

$$\int_{\mathbb{R}} f_n \, dx = 1 \neq \int 0 \, dx = 0$$

b) $f_n = n\,\chi_{[0,1/n]}$ $\qquad f_n \longrightarrow 0$ a.e

$$\int f_n \, dx = 1 \neq \int 0 \, dx = 0$$

PL : to show $\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$ we need to show 2 inequalities ; recall $f_n \in \mathcal{L}^+$ and $0 \leq f_n \leq f_{n+1} \leq \cdots \leq f$

f $f = \sup_{n\geq 1} f_n$ - What inequality is easier ?