Corollary: If $f, g : X \longrightarrow \overline{\mathbb{R}}$ are measurable. Then $\max\{f, g\}$ and $\min\{f, g\}$ measurable.

Remarks: ① $f : X \longrightarrow \overline{\mathbb{R}}$, $f^+(x) = \max\{0, f(x)\}$

$f^-(x) = \max\{0, -f(x)\}$

then $f = f^+ - f^-$ and $|f| = f^+ + f^-$

f is measurable iff f^+ & f^- measurable.

② $E \subset X$ $\quad \chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$ characteristic function of E

χ_E is measurable iff $E \in \mathcal{M}$

Pf: $\Longrightarrow$: $\chi_E^{-1}(\{1\}) = \{x : \chi_E(x) = 1\} = E \Rightarrow E \in \mathcal{M}$

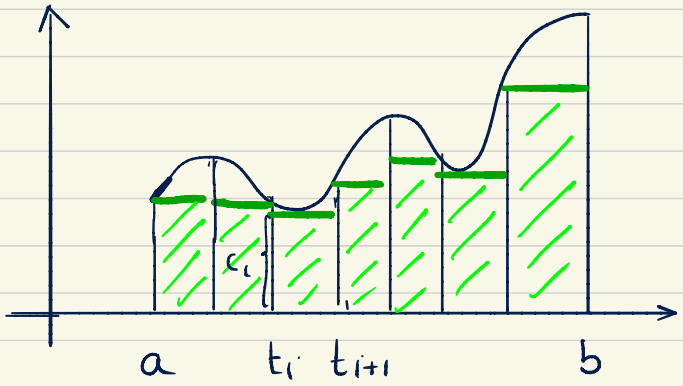
↑ Borel set

$\Longleftarrow$ to show χ_E meas $\{x : \chi_E(x) \leq \alpha\}$ is measurable for any α

$\{x : \chi_E(x) \leq \alpha\} = \begin{cases} \emptyset & \alpha < 0 \\ E^c & 0 \leq \alpha < 1 \\ X & \alpha \geq 1 \end{cases}$

χ_E is measurable

Riemann integration : Let $f \geq 0$ bounded piece wise continuous.



$$a = t_0 < t_1 < \cdots < t_n = b$$

$$E_i = [t_i, t_{i+1}) \qquad \Delta t_i = t_{i+1} - t_i$$

$$c_i = \min_{0 \leq i \leq n-1} \{ f(x) : x \in [t_i, t_{i+1}] \}$$

$$\sum_{i=0}^{n-1} c_i \, \Delta t_i = \sum_{i=0}^{n-1} c_i \, (t_{i+1} - t_i) = \sum_{i=0}^{n-1} c_i \, m E_i$$

$$m E_i = \int_a^b \chi_{E_i}(x) \, dx \qquad \psi_n(x) = \sum_{i=1}^{n-1} c_i \chi_{E_i}(x) \leq f$$

Q: What is the relationship between ψ_n & f ?

$$\left(\sup \psi_n(x) = f(x) \right.$$

$$\lim_{n \to \infty} \psi_n(x) = f(x)$$

Recall

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=0}^{n-1} c_i \, m E_i = \lim_{n \to \infty} \int_a^b \psi_n(x) \, dx$$ ////

Definition: A simple function $f : X \to \mathbb{R}$ is a finite linear combination of characteristic functions of measurable sets with real coefficients

$$f = \sum_{i=1}^{m} \alpha_i \chi_{A_i} \qquad A_i \in \mathcal{M}.$$

Remarks: If f, g are simple functions so are cf ; $f+g$; fg $(c \in \mathbb{R})$ $(\chi_A \chi_B = \chi_{A \cap B})$

We say that f is in standard representation if

$$f = \sum_{i=1}^{n} \beta_i \chi_{E_i} \qquad \text{where} \quad \beta_i \neq \beta_j$$

$$f^{-1}\{\beta_i\} = E_i \quad \text{and} \quad E_i \cap E_j = \emptyset$$

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Goals: ① Show that a measurable function can be approximated nicely by simple functions

② Define the integral for simple functions.

③ Extend definition to measurable functions.

Theorem (2.10) Let $(X, \mathcal{M})$ be a measurable space

a) If $f : X \longrightarrow [0, \infty]$ is measurable there is a sequence $\{\phi_n\}_n$ of simple functions s.t.

$$0 \leq \phi_1 \leq \phi_2 \leq \ldots \leq \phi_n \leq \ldots \leq f$$

with

$$\phi_n \xrightarrow[n \to \infty]{} f \quad \text{pointwise} \qquad f \quad \phi_n \xrightarrow[n \to \infty]{} f \quad \text{uniformly on sets on which } f \text{ is bounded}$$

b) If $f : X \longrightarrow \overline{\mathbb{R}}$ is measurable there is a sequence $\{\phi_n\}$ of simple functions s.t

$$0 \leq |\phi_1| \leq |\phi_2| \leq \ldots \leq |\phi_n| \leq \ldots \leq |f|$$

with

$$\phi_n \xrightarrow[n \to \infty]{} f \quad \text{pointwise} \qquad f \quad \phi_n \xrightarrow[n \to \infty]{} f \quad \text{uniformly on sets on which } f \text{ is bounded}$$

Proof : a) For $n = 0, 1, 2$ & $0 \le k \le 2^{2n} - 1$ let

$$E_n^k = f^{-1}\left(k2^{-n} ; (k+1)2^{-n}\right] \in \mathcal{M}$$

$$F_n = f^{-1}\left(2^n, \infty\right) \in \mathcal{M}$$

Define $$\phi_n = \sum_{k=0}^{2^{2n}-1} k2^{-n} \chi_{E_n^k} + 2^n \chi_{F_n}$$ simple function

Q: What is the main difference between this ϕ_n and the approximating ψ_n's in the Riemann integral ?

Proof : a) For $n = 0, 1, 2$ & $0 \le k \le 2^{2n} - 1$ let

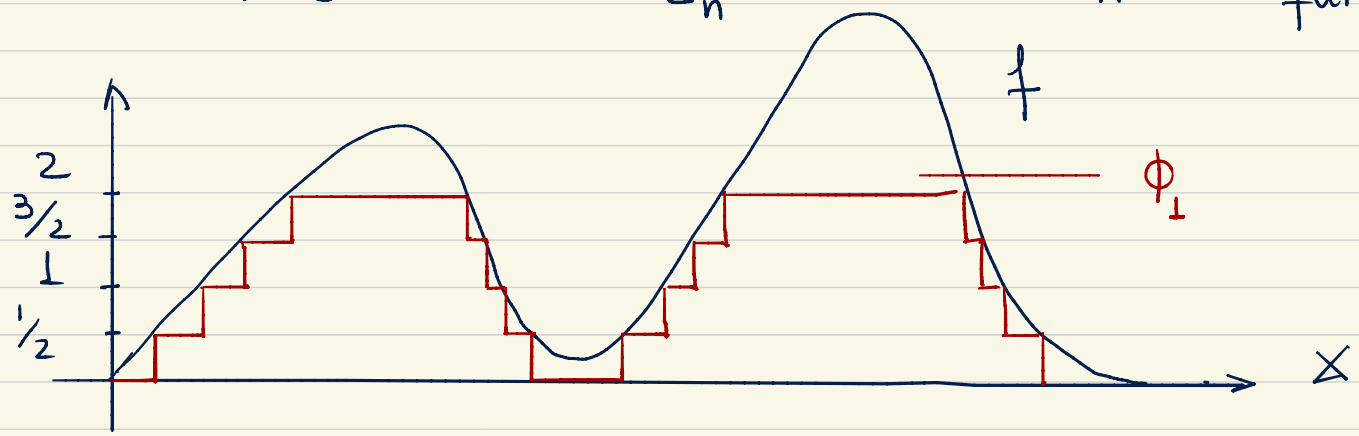
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Define $\phi_n = \sum_{k=0}^{2^{2n}-1} k2^{-n} \chi_{E_n^k} + 2^n \chi_{F_n}$ simple function

$n = 1$

$0 \le k \le 3$



Claim 1 : $\phi_n \le \phi_{n+1} \le f$

Claim 2 $0 \le f - \phi_n \le 2^{-n}$ on the set where $f \le 2^n$

Proof : a) For $n = 0, 1, 2$ & $0 \le k \le 2^{2n} - 1$ let

$$E_n^k = f^{-1}\left(k2^{-n} ; (k+1)2^{-n}\right] \qquad F_n = f^{-1}(2^n, \infty)$$

$$\phi_n = \sum_{k=0}^{2^{2n}-1} k2^{-n} \chi_{E_n^k} + 2^n \chi_{F_n}$$

Claim 1 : $\phi_n \le \phi_{n+1} \le f$ i) $f(x) = 0$ $\phi_n(x) = 0$

$f(x) > 0$, $n \in \mathbb{N}$ ① $\exists k \in \{0, \dots 2^{2n} - 1\}$ $f(x) \in \left(k2^{-n}, (k+1)2^{-n}\right]$

② $f(x) > 2^n \Rightarrow x \in F_n$ $\phi_n(x) = 2^n < f(x)$

① $f(x) > k2^{-n}$ & $x \in E_n^k$ $\phi_n(x) = k2^{-n} < f(x)$

$$x \in \left(k2^{-n}, (k+1)2^{-n}\right] = \underbrace{\left((2k)2^{-n-1}, (2k+1)2^{-n-1}\right]}_{2k} \cup \underbrace{\left((2k+1)2^{-n-1}, (2k+2)2^{-n-1}\right]}_{2k+1}$$

if $x \in E_{n+1}^{2k}$ if $x \in E_{n+1}^{2k+1}$

$\phi_n(x) = k2^{-n}$ $\phi_{n+1}(x) = 2k2^{-n-1} = k2^{-n}$ $\phi_{n+1}(x) = (2k+1)2^{-n-1}$

$$\phi_n(x) \le \phi_{n+1}(x)$$

if $x \in F_n$ $(\phi_n(x) = 2^n)$ $\quad f(x) \in (2^n, \infty)$

$f(x) \in (2^n, 2^{n+1}]$ $\quad f(x) \in (2^{n+1}, \infty)$

$x \in F_{n+1}$

$\phi_{n+1}(x) = 2^{n+1}$

$f(x) \in (\underbrace{2^{2(n+1)-1}}_{} 2^{-(n+1)}, 2^{2(n+1)} 2^{-n-1}]$

$\phi_{n+1}(x) = 2^{2(n+1)-1} = 2^{2n+1} > \phi_n(x)$

Claim 2 $\quad 0 \le f - \phi_n \le 2^{-n}$ $\quad$ if $\quad f(x) < 2^n$