Definition: If $(X, \mathcal{A})$ & $(Y, \mathcal{N})$ measurable spaces:

$f: X \to Y$ is $(\mathcal{A}, \mathcal{N})$ measurable if

$$\forall E \in \mathcal{N} \Rightarrow f^{-1}(E) \in \mathcal{A}.$$

Proposition: If $\mathcal{N}$ is a σ-algebra generated by $\mathcal{E}$, then $f: X \to Y$ is $(\mathcal{A}, \mathcal{N})$ measurable iff $f^{-1}(E) \in \mathcal{A}$ $\forall E \in \mathcal{E}$.

Corollary: If X & Y are metric spaces, every continuous function $f: X \to Y$ is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable.

Pf of proposition: $\boxed{\Rightarrow}$ ✓

$\boxed{\Leftarrow}$ $\mathcal{O} = \{E \subset Y : f^{-1}(E) \in \mathcal{A}\}$, $\mathcal{E} \subset \mathcal{O}$

Goal: show $\mathcal{O}$ is a σ-algebra $\Rightarrow$ $\mathcal{N} \subset \mathcal{O}$

($\mathcal{N}$: smallest σ-algebra containing $\mathcal{E}$)

$\mathcal{C} = \{E \subset \mathcal{U} : f^{-1}(E) \in \mathcal{A}\}$

① $\phi \in \mathcal{C}$ $\quad f^{-1}(\phi) = \phi \in \mathcal{A}$ ← σ-algebra

② $E \in \mathcal{C}$ (want $E^c \in \mathcal{C}$) $\quad f^{-1}(E^c) = (f^{-1}(E))^c$

$f^{-1}(E) \in \mathcal{A} \underset{\sigma\text{-alg}}{\Rightarrow} (f^{-1}(E))^c \in \mathcal{A} \longrightarrow f^{-1}(E^c) \in \mathcal{A}$

$\Downarrow E^c \in \mathcal{C}$

③ $\{E_i\}_i \subset \mathcal{C}$ $\quad f^{-1}(E_i) \in \mathcal{A}$

$$\bigcup_i f^{-1}(E_i) = f^{-1}\left(\bigcup_i E_i\right) \in \mathcal{A}$$

$$\Rightarrow \bigcup E_i \in \mathcal{C}$$

Remark: $\mathcal{B}_{\mathbb{R}}$ is generated by:

i) $\{(a,b)\}_{a<b}$; ii) $\{(-\infty,a)\}_{a\in\mathbb{R}}$; iii) $\{(-\infty,a]\}_{a\in\mathbb{R}}$

iv) $\{(a,\infty)\}_{a\in\mathbb{R}}$; v) $\{[a,\infty)\}_{a\in\mathbb{R}}$

Pf: for i) & ii)

every open set in $\mathbb{R}$ is a countable union of (disjoint) intervals of type i)

ii) $$[a,b) = (-\infty,b) \cap ((-\infty,a))^c$$

$$(a,b) = \bigcup_{n=n_0}^{\infty} [a+\frac{1}{n}\,;\,b)$$ n_0 big enough.

Remark: $\mathcal{B}_{\mathbb{R}}$ is generated by:

i) $\{(a,b)\}_{a<b}$; ii) $\{(-\infty,a)\}_{a\in\mathbb{R}}$; iii) $\{(-\infty,a]\}_{a\in\mathbb{R}}$

iv) $\{(a,\infty)\}_{a\in\mathbb{R}}$; v) $\{[a,\infty)\}_{a\in\mathbb{R}}$

Proposition: If $(X,\mathcal{M})$ measurable space & $f: X \to \mathbb{R}$ t.f.a.e.

i) f is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ measurable

ii) $f^{-1}(a,\infty) \in \mathcal{M}$ $\forall a \in \mathbb{R}$

iii) $f^{-1}(-\infty, a) \in \mathcal{M}$ $\forall a \in \mathbb{R}$

iv) $f^{-1}(-\infty, a] \in \mathcal{M}$ $\forall a \in \mathbb{R}$

v) $f^{-1}[a,\infty) \in \mathcal{M}$ $\forall a \in \mathbb{R}$

vi) $f^{-1}(a,b) \in \mathcal{M}$ $\forall a, \forall b \in \mathbb{R}$

Terminology: 1) If $(X, \mathcal{M})$ is a measurable space; $f: X \to \mathbb{R}$ is **measurable** if it is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ measurable

Lebesgue σ-algebra (m_*)

2) $f: \mathbb{R} \to \overline{\mathbb{R}}$ is **Lebesgue measurable** if it is $(\mathcal{L}, \mathcal{B}_{\mathbb{R}})$-measurable

Borel measurable if it is $(\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}})$-measurable

3) $f, g: \mathbb{R} \to \mathbb{R}$ Lebesgue measurable $f \circ g$ is NOT in general Lebesgue measurable. (not even if g is continuous)

$f: (\mathbb{R}, \mathcal{L}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ $\quad$ $g: (\mathbb{R}, \mathcal{L}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

Q: What is the issue? $\quad f \circ g: \mathbb{R} \to \mathbb{R}$

$E \in \mathcal{B}_{\mathbb{R}}$ $\quad (f \circ g)^{-1}(E) = g^{-1}(f^{-1}(E))$ $\quad$ possible issue

$f^{-1}(E) \in \mathcal{L}$ (might not Borel)

f continuous

g measurable $\quad U \subset \mathbb{R}$ open

$g^{-1}(f^{-1}(U)) \in \mathcal{L}$

$f^{-1}(U)$: open

Terminology: 1) If $(X, \mathcal{M})$ is a measurable space; $f: X \to \mathbb{R}$ is **measurable** if it is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ measurable

2) $f: \mathbb{R} \to \mathbb{R}$ is **Lebesgue measurable** if it is $(\mathcal{L}, \mathcal{B}_{\mathbb{R}})$-measurable

Borel measurable if it is $(\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}})$-measurable

3) $f, g: \mathbb{R} \to \mathbb{R}$ Lebesgue measurable $f \circ g$ is NOT in general Lebesgue measurable. (not even if g is continuous)

$f: (\mathbb{R}, \mathcal{L}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ $\quad$ $g: (\mathbb{R}, \mathcal{L}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

4) If $f, g: \mathbb{R} \to \mathbb{R}$ / f Borel measurable & g Lebesgue meas.

then $f \circ g$ is Lebesgue measurable

$$(\mathbb{R}, \mathcal{L}) \xrightarrow{g} (\mathbb{R}, \mathcal{B}_{\mathbb{R}}) \xrightarrow{f} (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$$

$g^{-1}(f^{-1}(E)) \in \mathcal{L}$ $\quad$ $f^{-1}(E) \in \mathcal{B}_{\mathbb{R}}$ $\quad$ $E \in \mathcal{B}_{\mathbb{R}}$

Proposition: $(X, \mathcal{M})$ measurable space, $f, g : X \to \mathbb{R}$ measurable, $c \in \mathbb{R}$. Then cf, $f+g$, fg are measurable

Pf: is cf measurable : $\alpha \in \mathbb{R}$

$c \neq 0$ $\quad \{x \in X : cf(x) < \alpha\} = \{x \in X : f(x) < \alpha/c\} \in \mathcal{M}$

($\uparrow$ $c > 0$; f measurable)

$= \{x \in X : f(x) > \alpha/c\} \in \mathcal{M}$

($\uparrow$ $c < 0$)

$c = 0$ $\quad \{x \in X : 0 \cdot f(x) < \alpha\} = \begin{cases} X & \text{if } \alpha > 0 \\ \emptyset & \text{if } \alpha \leq 0 \end{cases}$

ii) $A_\alpha = \{x \in X : f(x) + g(x) < \alpha\}$ $\quad H : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \to x + y$

measurable (?)

$G : X \to \mathbb{R}^2 \to \mathbb{R}$; H continuous

$x \mapsto (f(x), g(x)) \to f(x) + g(x)$

$x \in A_\alpha$ iff $f(x) < \alpha - g(x)$ $\exists r \in \mathbb{Q}$

$f(x) < r < \alpha - g(x)$

$x \in A_\alpha \Rightarrow \{x \in X : f(x) < r\} \cap \{x : g(x) < \alpha - r\}$

$$A_\alpha = \bigcup_{r \in \mathbb{Q}} \underbrace{\{f(x) < r\}}_{\in \mathcal{M}} \cap \underbrace{\{x : g(x) < \alpha - r\}}_{\in \mathcal{M}} \in \mathcal{M}.$$

iii) f, g measurable $\Rightarrow f+g$, $f-g$ measurable

$$fg = \frac{1}{4}\{(f+g)^2 - (f-g)^2\}$$

enough to show

if h measurable then h^2 is measurable

$$\{x : h^2(x) > \alpha\} =$$

$\alpha \geq 0$

Consider $\overline{\mathbb{R}} = [-\infty, \infty]$ $\quad \mathcal{B}_{\overline{\mathbb{R}}} = \{E \subset \overline{\mathbb{R}} : E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}$

note $\mathcal{B}_{\overline{\mathbb{R}}}$ is generated by $\{(a, \infty]\}_{a \in \mathbb{R}} \cup \{[-\infty, a)\}_{a \in \mathbb{R}}$

$f: (X, \mathcal{M}) \longrightarrow (\overline{\mathbb{R}}, \mathcal{B}_{\overline{\mathbb{R}}})$ is measurable $\#$

it is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$ measurable

Convention $\quad 0 \cdot \infty = 0$ $\quad$ careful with $\infty - \infty$

Proposition: $(X, \mathcal{M})$ measurable space, $f, g : X \to \overline{\mathbb{R}}$ measurable

$c \in \mathbb{R}$ - Then cf, $f+g$, fg are measurable (Pb 2 Ch 2)

Proposition: $(X, \mathcal{M})$ measurable space; $f_j : X \longrightarrow \overline{\mathbb{R}}$ measurable $j \in \mathbb{N}$

then

$$g_1(x) = \sup_j f_j(x) \qquad g_3(x) = \limsup_{j \to \infty} f_j(x)$$

$$g_2(x) = \inf_j f_j(x) \qquad g_4(x) = \liminf_{j \to \infty} f_j(x)$$

are all measurable functions from X to $\overline{\mathbb{R}}$.

If $f(x) = \lim_{j \to \infty} f_j(x)$ exists $\forall x \in X$ then f is measurable.

$f_j : X \to \mathbb{R}$ $\qquad g_1(x) = \sup_j f_j(x)$ is measurable

$$\{ g_1(x) > \alpha \} = \{ \sup_j f_j(x) > \alpha \}$$

$$= \bigcup_j \underbrace{\{ f_j(x) > \alpha \}}_{\in \mathcal{M}} \in \mathcal{M}$$

$$g_2(x) = \inf_j f_j(x) = - \sup_j - f_j(x)$$

$$g_3(x) = \limsup f_j = \inf_j \sup_{k \geq j} f_k(x)$$

$$g_4(x) = \liminf f_j = \sup_j \inf_{k \geq j} f_k(x)$$