Terminology

$X \neq \emptyset$

μ outer measure $\begin{cases} \mu^* : 2^X \longrightarrow [0, \infty] \text{ s.t } \mu^*(\emptyset) = 0 \\ \mu^* \text{ subadditive} \end{cases}$

$\mathcal{M}$ σ-algebra

μ measure $\begin{cases} \mu : \mathcal{M} \longrightarrow [0, \infty] \text{ outer measure } \& \\ \mu \text{ countably additive} \end{cases}$

Example : $\mathcal{M} = \{ E \subset 2^X ; \mu^* \text{ measurable} \}$, μ^* outer measure

$\mu = \mu^*|_{\mathcal{M}}$ measure

$(X, \mathcal{M}, \mu)$ measure space

$(X, \mathcal{M})$ measurable space

Proposition: Let (X, ρ) be a metric space, μ be a Borel measure on X.

Assume

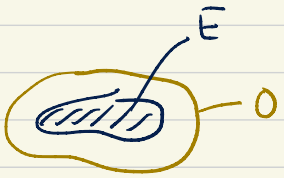
$$\mu(B(x,r)) < \infty \qquad \forall x \in X \qquad \forall r > 0$$

Then

(*) for any $E \in \mathcal{B}_X$, $\forall \varepsilon > 0$ there exists an open set $\mathcal{O}$ and a closed set F such that

$$F \subset E \subset \mathcal{O} \quad \text{and}$$

$$\mu(\mathcal{O} \setminus E) < \varepsilon \quad \& \quad \mu(E \setminus F) < \varepsilon$$



A Borel measure μ satisfying (*) is called a Borel regular measure

To show that $\mathcal{B}_X = \mathcal{C}$ we need

$$\mathcal{C} = \{ E \in \mathcal{B}_X : (*) \text{ holds} \} \subset \mathcal{B}_X$$

Claim 1: $\mathcal{C}$ is a σ-algebra

Claim 2 $\mathcal{C}$ contains all open sets (thus it contains the σ-algebra generated by the open sets $\mathcal{B}_X$) - Hence $\mathcal{B}_X \subset \mathcal{C} \Rightarrow \mathcal{B}_X = \mathcal{C}$

$\mathcal{C} = \{ E \in \mathcal{B}_X : \forall \varepsilon > 0 \ \exists O_\varepsilon \text{ open } \& \ \exists F_\varepsilon \text{ closed s.t}$

$\circledast \ \{ F_\varepsilon \subset E \subset O_\varepsilon , \ \mu(O_\varepsilon \setminus E) < \varepsilon \ \& \ \mu(E \setminus F_\varepsilon) < \varepsilon \}$

is a σ-algebra : ① $X, \phi \in \mathcal{C}$

(we want $E^c \in \mathcal{C}$)

② $E \in \mathcal{C}$ $\circledast$ holds ; note $O_\varepsilon^c \subset E^c \subset F_\varepsilon^c$ (O_ε^c closed, F_ε^c open)

$O_\varepsilon \setminus E = O_\varepsilon \cap E^c = E^c \cap (O_\varepsilon^c)^c = E^c \setminus O_\varepsilon^c$ (O_ε^c closed)

$E \setminus F_\varepsilon = E \cap F_\varepsilon^c = F_\varepsilon^c \cap (E^c)^c = F_\varepsilon^c \setminus E^c$ (F_ε^c open)

$\& \ \mu(F_\varepsilon^c \setminus E^c) < \varepsilon$

$\& \ \mu(E^c \setminus O_\varepsilon^c) < \varepsilon \implies E^c \in \mathcal{C}$

$\mathcal{C} = \{ E \in \mathcal{B}_X : \forall \varepsilon > 0 \ \exists O_\varepsilon \text{ open } \& \ \exists F_\varepsilon \text{ closed s.t}$

(*) $\{ F_\varepsilon \subset E \subset O_\varepsilon , \ \mu(O_\varepsilon \setminus E) < \varepsilon \ \& \ \mu(E \setminus F_\varepsilon) < \varepsilon \}$

is a σ-algebra : (1) $X, \phi \in \mathcal{C}$

(2) $E \in \mathcal{C}$ (*) holds ; note $O_\varepsilon^c \subset E^c \subset F_\varepsilon^c$

$O_\varepsilon \setminus E = O_\varepsilon \cap E^c = E^c \cap (O_\varepsilon^c)^c = E^c \setminus O_\varepsilon^c$ $\quad O_\varepsilon^c$ closed

$E \setminus F_\varepsilon = E \cap F_\varepsilon^c = F_\varepsilon^c \cap (E^c)^c = F_\varepsilon^c \setminus E^c$ $\quad F_\varepsilon^c$ open

$\& \quad \mu(F_\varepsilon^c \setminus E^c) < \varepsilon$

$\& \quad \mu(E^c \setminus O_\varepsilon^c) < \varepsilon \implies E^c \in \mathcal{C}$

(3) $\{E_k\} \subset \mathcal{C}$ $\quad E = \bigcup_{k=1}^{\infty} E_k$ $\quad \forall \varepsilon > 0 \ \ \forall k \in \mathbb{N}$

$\exists \ \widetilde{F}_k \subset E_k \subset O_k$ s.t $\quad \mu(O_k \setminus E_k) < \varepsilon / 2^k$

$\widetilde{F}_k$ closed, O_k open

$\mu(E_k \setminus \widetilde{F}_k) < \varepsilon / 2^k$

$$\mathcal{O} = \bigcup_{k=1}^{\infty} O_k \quad \text{open} \quad E \subset \mathcal{O} \qquad \mathcal{O} \setminus E = \bigcup_{k=1}^{\infty} O_k \setminus \left(\bigcup_{k=1}^{\infty} E_k \right)$$

$$\mathcal{O} \setminus E = \bigcup_{k=1}^{\infty} O_k \cap \left(\bigcup_{k=1}^{\infty} E_k \right)^c = \bigcup_{k=1}^{\infty} O_k \cap \left(\bigcap_{j=1}^{\infty} E_j^c \right)$$

$$\subseteq \bigcup_{k=1}^{\infty} O_k \cap E_k^c = \bigcup_{k=1}^{\infty} O_k \setminus E_k$$

$$\mu(\mathcal{O} \setminus E) \leq \sum_{k=1}^{\infty} \mu(O_k \setminus E_k) \leq \varepsilon$$

($\mu(O_k \setminus E_k) < \varepsilon/2^k$)

let $F^* = \bigcup_{k=1}^{\infty} \widetilde{F}_k$ (not necessarily closed!) nevertheless

as before $\mu(E \setminus F^*) \leq \varepsilon$ (Note we have not used the hypothesis!)

let $F_k = \bigcup_{j=1}^{k} \widetilde{F}_j$ closed sets $F_k \subset F_{k+1}$

Claim: given $\varepsilon > 0$ $\exists F_\varepsilon$ closed s.t. $F_\varepsilon \subset F^*$ & $\mu(F^* \setminus F_\varepsilon) < \varepsilon$

$\mu(E \setminus F_\varepsilon) \leq \mu(E \setminus F^*) + \mu(F^* \setminus F_\varepsilon) \leq 2\varepsilon$ $\rightsquigarrow$ $\mathcal{C}$ closed under countable unions

Idea: look at $\mu(F^* \setminus F_n)$ $\qquad F^* = \bigcup_{k=1}^{\infty} \tilde{F}_k = \bigcup_{k=1}^{\infty} F_k$

since $F_n \subset F_{n+1} \Rightarrow F^* \setminus F_{n+1} \subset F^* \setminus F_n$

we would like to say

"$$\lim_{n \to \infty} \mu(F^* \setminus F_n) = \mu\left(\bigcap_{n=1}^{\infty} F^* \setminus F_n\right) = \mu\left(\bigcap_{n=1}^{\infty} F^* \cap F_n^c\right)$$"

$$\bigcap_{n=1}^{\infty} F^* \cap F_n^c = F^* \cap \bigcap_{n=1}^{\infty} F_n^c = F^* \cap \left(\bigcup_{n=1}^{\infty} F_n\right)^c = \emptyset$$

thus $\lim_{n \to \infty} \mu(F^* \setminus F_n) = 0 \qquad \forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N}$ s.t

$$\mu(F^* \setminus F_{n_\varepsilon}) < \varepsilon$$

Is this argument correct ?

Issue : $\mu(F^* \setminus F_1)$ might be infinity.

Q: How do we deal with this issue? How do we use the hypothesis?

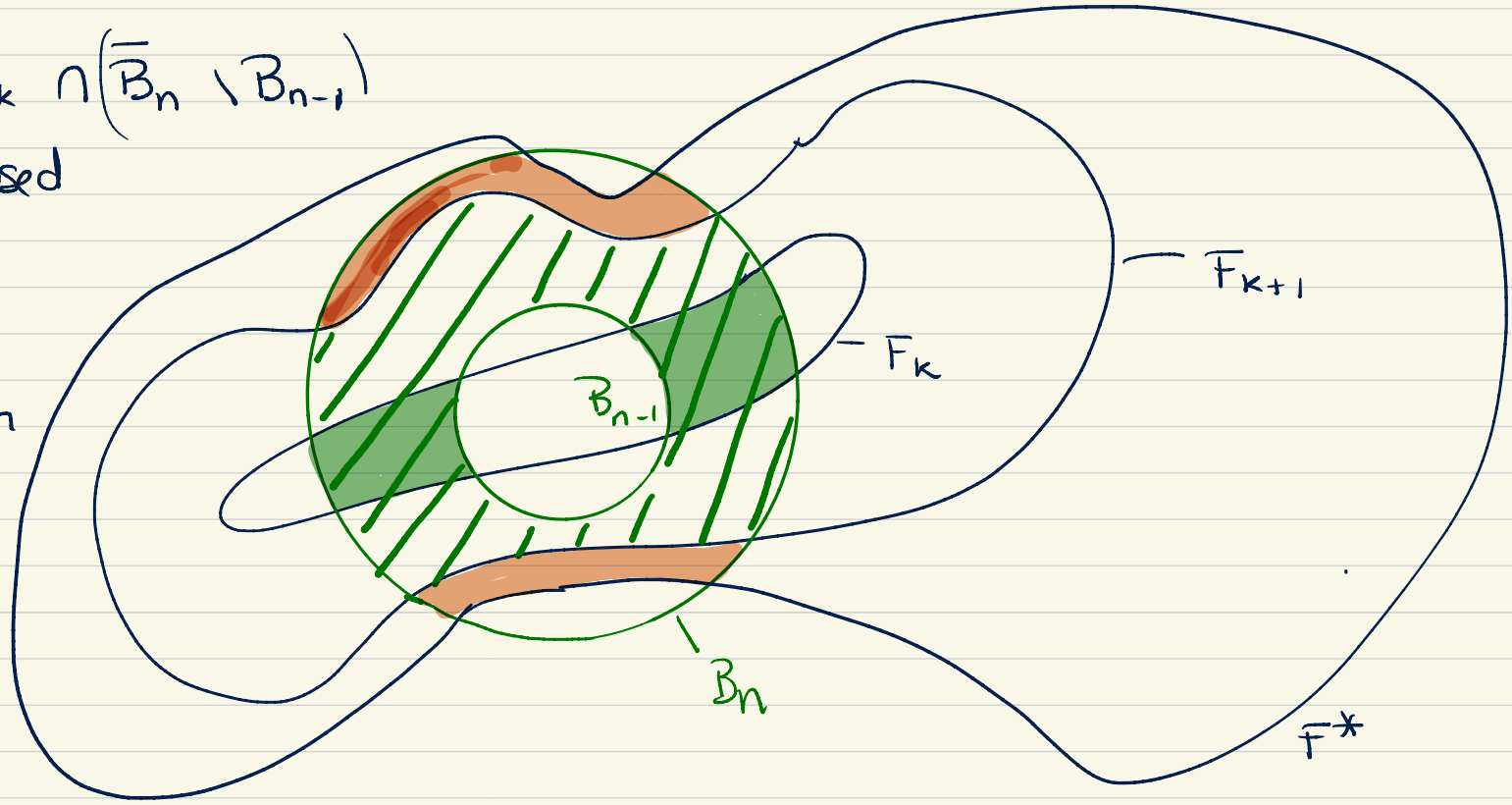
$x_0 \in X$ $\quad B_n = B(x_0, n)$ open ball $\quad B_0 = \phi \quad X = \bigcup_n B_n$

$\mu B_n < \infty \quad \mu$ - σ - finite. Recall $F_k \subset F_{k+1}$

$G_{k,n} = F_k \cap (\overline{B}_n \setminus B_{n-1})$ closed

$G_{k+1,n}$

$G_{k,n} \subset G_{k+1,n}$



$H_{k,n} = (F^* \cap \overline{B}_n \setminus B_{n-1}) \setminus G_{n,k} \qquad H_{k+1,n} \subset H_{k,n}$

$\mu(H_{k,n}) \leq \mu(B_{2n}) < \infty$

Claim: For each n $\quad \mu(H_{k,n}) \longrightarrow 0$ as $k \to \infty$

Thus $\exists N(n)$ s.t. $\mu(H_{N(n),n}) < \varepsilon/2^n$

$$H_{N(n),n} = F^* \cap (\overline{B}_n \setminus B_{n-1}) \setminus G_{N(n),n}$$

$$= (F^* \setminus F_{N(n)}) \cap \overline{B}_n \setminus B_{n-1}$$

(★) $F = \bigcup_{n=1}^{\infty} F_{N(n)} \cap (\overline{B}_n \setminus B_{n-1})$ since $F_{N(n)} \subset F^*$

$$F^* \setminus F \subset \bigcup_{n=1}^{\infty} (F^* \setminus F_{N(n)}) \cap \overline{B}_n \setminus B_{n-1} = \bigcup_{n=1}^{\infty} H_{N(n),n}$$

$$\mu(F^* \setminus F) \leq \sum_{n=1}^{\infty} \mu(H_{N(n),n}) < \varepsilon$$

Q: What are we missing?

Claim $F = \bigcup_{n=1}^{\infty} F_{N(n)} \cap \overline{B}_n \setminus B_{n-1}$ is closed

let $\{x_j\} \subset F$ $\quad x_j \longrightarrow x$ $\quad \exists n$ s.t $d(x_0, x) < n+1$

$\exists j_0 \in \mathbb{N}$ $\quad \forall j \geq j_0$ $\quad d(x_0, x_j) < n+1$

$$x_j \in \bigcup_{k=1}^{n+1} \underbrace{F_{N(k)} \cap \overline{B}_k \setminus B_{k-1}}_{\text{closed}} \qquad \text{closed}$$

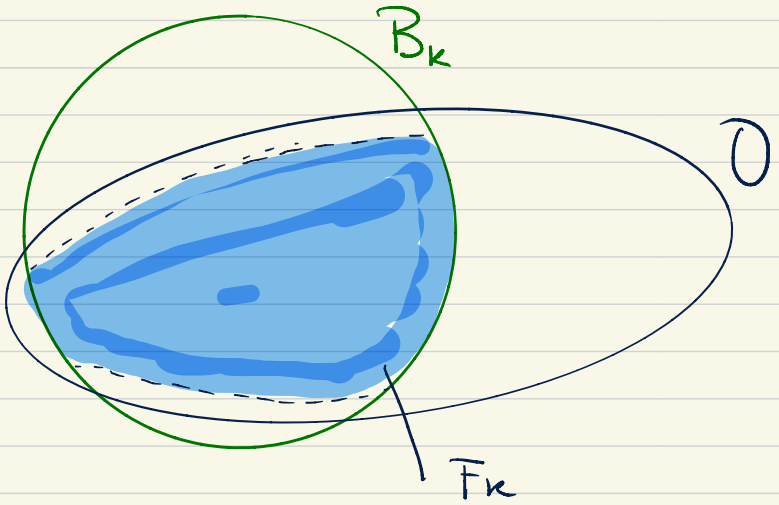
$$x_j \longrightarrow x \in \bigcup_{k=1}^{n+1} F_{N(k)} \cap (\overline{B}_k \setminus B_{k-1}) \subset F$$

$\Rightarrow$ F closed

This concludes the proof of Claim 1

Claim 2 : O open then $O \in \mathcal{C}$

let $F_k = \{ x \in \overline{B}_k : d(x, O^c) \geq 1/k \}$



Claims

(1) F_k closed

(2) $O = \bigcup_{k=1}^{\infty} F_k$

Apply previous argument to conclude

$\forall \varepsilon > 0 \quad \exists F_\varepsilon$ closed such that

$F_\varepsilon \subset O$ & $\mu(O \setminus F_\varepsilon) < \varepsilon$

$\mathcal{C}$ σ-algebra contains open sets then $\mathcal{B}_X \subset \mathcal{C}$ ∎

Measurable functions

Recall $f: X \to Y$; $f^{-1}: 2^Y \to 2^X$ $\quad E \subset Y$

$$f^{-1}(E) = \{ x \in X : f(x) = y : y \in E \}$$

Properties: f^{-1} preserves unions, complements, intersections:

i) $f^{-1}\left(\bigcup_\alpha A_\alpha\right) = \bigcup_\alpha f^{-1}(A_\alpha)$

2) $f^{-1}\left(\bigcap_\alpha A_\alpha\right) = \bigcap_\alpha f^{-1}(A_\alpha)$

3) $f^{-1}(A^c) = [f^{-1}(A)]^c$

Thus if $\mathcal{N}$ σ-algebra in Y,

$\{ f^{-1}(E) : E \in \mathcal{N} \}$ is a σ-algebra in X.

Definition: If $(X, \mathcal{M})$ & $(Y, \mathcal{N})$ measurable spaces:

$f: X \to Y$ is $(\mathcal{M}, \mathcal{N})$ measurable if

$\forall E \in \mathcal{N} \Rightarrow f^{-1}(E) \in \mathcal{M}$.

Remark: if $f:(X,\mathcal{M}) \to (Y,\mathcal{N})$ measurable and $g:(Y,\mathcal{N}) \to (Z,\mathcal{O})$ measurable

then

$g \circ f : (X,\mathcal{M}) \to (Z,\mathcal{O})$ measurable.

① Is $f:(X,2^X) \to (Y,\mathcal{N})$ measurable? Yes

② Is $f:(\mathbb{R},\mathcal{B}_{\mathbb{R}}) \to (\mathbb{R},\mathcal{B}_{\mathbb{R}})$ continuous, measurable?

Yes, because if $U \subset \mathbb{R}$ open then

$f^{-1}(U)$ is open

($\mathcal{E}$ = {open sets} $\quad \sigma(\mathcal{E}) = \mathcal{B}_{\mathbb{R}}$)

Proposition: If $\mathcal{N}$ is a σ-algebra generated by $\mathcal{E}$, then $f: X \to Y$ is $(\mathcal{M}, \mathcal{N})$ measurable iff $f^{-1}(E) \in \mathcal{M}$ $\forall E \in \mathcal{E}$.