Definition : Let (X, ρ) be a metric space, an outer (exterior) measure μ_* on X is a metric outer (exterior) measure if

$$\mu_*(A \cup B) = \mu_*(A) + \mu_*(B) \qquad (*)$$

whenever

$$d(A,B) = \inf_{\substack{a \in A \\ b \in B}} \rho(a,b) > 0$$

Lemma : Let μ_* be a metric outer measure on (X, ρ), then all closed sets are μ_*-measurable, thus so are all open sets, and μ_* is Borel

Pf : Let $F \subset X$ closed set, $E \subset X$ if $\mu_* E = +\infty$ then

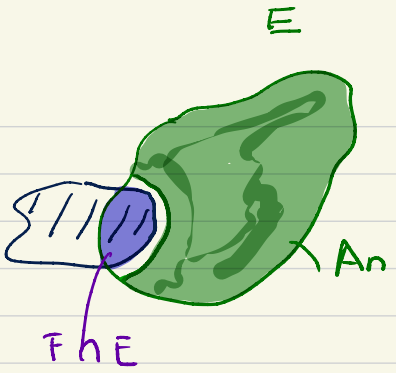
$$\mu_* E \geq \mu_*(F \cap E) + \mu_*(E \cap F^c)$$

Assume $\mu_* E < \infty$ - For $n \in \mathbb{N}$ consider

$$A_n = \left\{ x \in F^c \cap E : d(x, F) \geq \frac{1}{n} \right\}$$

Assume $\mu^* E < \infty$ - For $n \in \mathbb{N}$ consider

$$A_n = \{x \in F^c \cap E : d(x, F) \geq \frac{1}{n}\}$$



Claim 1: $A_n \subset A_{n+1}$ & $\bigcup_{n=1}^{\infty} A_n = F^c \cap E$ & $d(F \cap E, A_n) \geq \frac{1}{n}$

- $A_n \subset F^c \cap E \quad \forall n \Rightarrow \bigcup_{n=1}^{\infty} A_n \subset F^c \cap E$
- if $x \in F^c \cap E$, $x \in F^c$ open $\exists n \in \mathbb{N}$ s.t $B(x, \frac{1}{n}) \subset F^c$

  $\Rightarrow d(x, F) \geq \frac{1}{n}$ ie $x \in A_n$ $F^c \cap E \subset \bigcup_{n=1}^{\infty} A_n$
- if $x \in F \cap E$ and $y \in A_n$ $\rho(y, x) \geq \frac{1}{n}$ by the definition of A_n. Thus

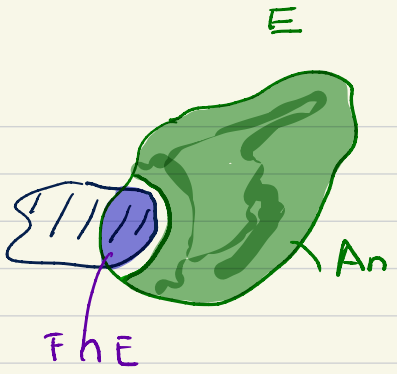
  $$d(F \cap E, A_n) \geq 1/n$$

Since μ^* is a metric outer measure, for any $n \in \mathbb{N}$

$$\mu^* E \geq \mu^*(E \cap F \cup A_n) \geq \mu^*(E \cap F) + \mu^*(A_n)$$

Assume $\mu^* E < \infty$ - For $n \in \mathbb{N}$ consider

$$A_n = \{x \in F^c \cap E : d(x, F) \geq \frac{1}{n}\}$$



<u>Claim 1</u>: $A_n \subset A_{n+1}$ & $\bigcup_{n=1}^{\infty} A_n = F^c \cap E$ & $d(F \cap E, A_n) \geq \frac{1}{n}$

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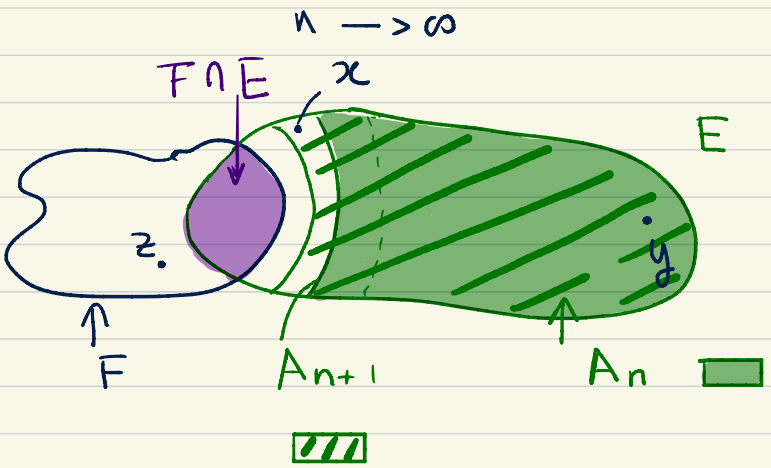
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  $$d(F \cap E, A_n) \geq 1/n$$

Since μ^* is a metric outer measure, for any $n \in \mathbb{N}$

$$\mu^* E \geq \mu^*(E \cap F \cup A_n) \geq \mu^*(E \cap F) + \mu^*(A_n)$$

Claim 2 : $\lim_{n\to\infty} \mu_*(A_n) = \mu_*(F^c \cap E)$



$B_n = A_{n+1} \cap A_n^c$

if $x \in B_n$

$\frac{1}{n+1} \le d(x, F) < \frac{1}{n}$

Consider $x \in B_n$, $y \in A_{n-1}$ $n \ge 2$

$z \in F$

$$\rho(x,y) \ge \rho(y,z) - \rho(z,x) \ge \frac{1}{n-1} - \frac{1}{n} = \frac{1}{n(n-1)}$$

$$d(B_n, A_{n-1}) \ge \frac{1}{n(n-1)} > 0$$

$$B_{2k} = A_{2k+1} \cap A_{2k}^c \subset A_{2k+1} \qquad A_{2k-1} \subset A_{2k+1}$$

$$\mu_*(A_{2k+1}) \ge \mu_*(A_{2k-1} \cup B_{2k}) \ge \mu_*(A_{2k-1}) + \mu_*(B_{2k})$$

$d(B_{2k}, A_{2k-1}) > 0$ ↑ by (*)

similarly

$$\mu_*(A_{2k}) \ge \mu_*(A_{2k-2}) + \mu_*(B_{2k-1})$$

$A_{2k-2}, B_{2k-1} \subset A_{2k}$ & $d(A_{2k-2}, B_{2k-1}) > 0$

$$\infty > \mu_*(E) \geq \mu_*(A_{2k+1}) \geq \mu_*(A_{2k-1}) + \mu_*(B_{2k})$$

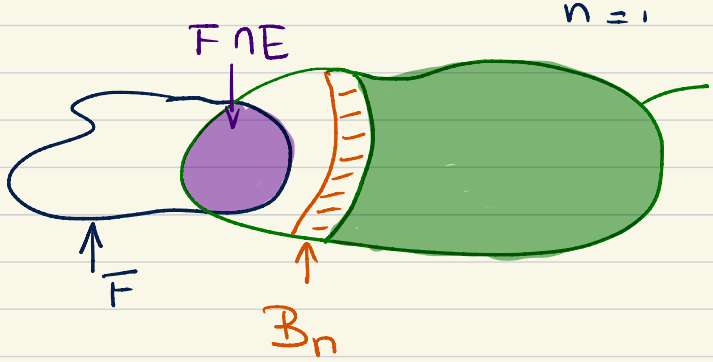
$$\geq \mu_*(A_{2k-3}) + \mu_*(B_{2k-2}) + \mu_*(B_{2k})$$

$$\geq \sum_{j=1}^{k} \mu_*(B_{2j})$$

Similarly $\infty > \mu_*(E) \geq \mu_*(A_{2k}) \geq \sum_{j=1}^{k} \mu_*(B_{2j-1})$

$\Rightarrow \sum_{j=1}^{\infty} \mu_*(B_j) < \infty$ converges

Since $F^c \cap E = \bigcup_{n=1}^{\infty} A_n = A_n \cup \bigcup_{j=n}^{\infty} B_j$ by



monotonicity and subadditivity

$$\mu_* A_n \leq \mu_*(F^c \cap E)$$

$$\leq \mu_*(A_n) + \mu_*\left(\bigcup_{j=n}^{\infty} B_j\right)$$

$$\mu_* A_n \leq \mu_*(F^c \cap E) \leq \mu_*(A_n) + \sum_{j=n}^{\infty} \mu_*(B_j)$$

letting $n \to \infty$ we have

$$\mu_*(F^c \cap E) = \lim_{n \to \infty} \mu_*(A_n)$$

Measures

(1) Given $X \neq \phi$, a σ-algebra $\mathcal{M} \subset 2^X$ and an outer measure $\mu : \mathcal{M} \longrightarrow [0, \infty]$, we say that μ is a **measure** if given any collection $\{E_i\}_{i=1}^{\infty} \subset \mathcal{M}$ of disjoint sets

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$ (countable additivity)

(2) $(X, \mathcal{M}, \mu)$ is a **measure space**

(3) $(X, \mathcal{M}, \mu)$ is a **complete** measure space if whenever $F \in \mathcal{M}$, $\mu F = 0$ & $E \subset F$ then $E \in \mathcal{M}$.

(4) $(X, \mathcal{M}, \mu)$ is **σ-finite** if $X = \bigcup_{j=1}^{\infty} X_j$, $X_j \in \mathcal{M}$ and

$$\mu X_j < \infty$$

Example : $X \neq \phi$, μ an outer measure ,

$\mathcal{M} = \{ E \subset X : E \ \mu\text{-measurable} \}$ is σ-algebra & μ-countably additive (property ii) of Theorem 10-09-2020)

Thus $\mu : \mathcal{M} \to [0, \infty]$ is a measure, $(X, \mathcal{M}, \mu)$ a measure space.

If $F \in \mathcal{M}$, $\mu F = 0$ & $E \subset F$ by monotonicity $\mu E = 0$ and all sets all μ-measure 0 are μ-measurable thus $E \in \mathcal{M}$. Hence $(X, \mathcal{M}, \mu)$ is a complete measure space.

Theorem : Given an outer measure μ_* on $X \neq \phi$, the collection $\mathcal{M}$ of μ_* measurable (Caratheodory measurable) sets is a σ-algebra. Moreover $\mu_* | \mathcal{M} = \mu$ (μ_* restricted to $\mathcal{M}$) is a measure and

$(X, \mathcal{M}, \mu)$ is a complete measure space.

Example : m_* outer measure in $\mathbb{R}^d$

$$\mathcal{L} = \{ E \subset \mathbb{R}^d : E \text{ is } m_* \text{ measurable} \}$$

$m = m_*|_{\mathcal{L}} : \mathcal{L} \longrightarrow [0,\infty]$ is a measure and

$(\mathbb{R}^d, \mathcal{L}, m)$ is a complete measure space, m Lebesgue measure.

Moreover all open sets are m_* measurable, so the Borel σ-algebra $\mathcal{B}_{\mathbb{R}^d}$ (smallest σ-algebra containing all open sets) so

$$\mathcal{B}_{\mathbb{R}^d} \subset \mathcal{L}$$

Question : Are $\mathcal{B}_{\mathbb{R}^d}$ and $\mathcal{L}$ different ?

Proposition : If μ_* on (X, ρ) is a metric outer measure then $\mu = \mu_*|_{\mathcal{B}_X}$ is a measure and $\mu = \mu_*|_{\mathcal{M}}$ ($\mathcal{M}$ = collection of μ_*-measurable) is a complete measure

$$\mathcal{B}_X \subset \mathcal{M}.$$

Proposition: let (X, ρ) be a metric space, μ be a Borel measure on X.

Assume

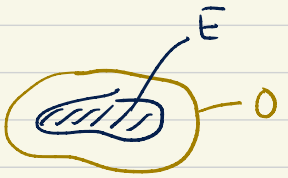
$$\mu(B(x,r)) < \infty \qquad \forall x \in X \qquad \forall r > 0$$

Then

(*) for any $E \in \mathcal{B}_X$, $\forall \varepsilon > 0$ there exists an open set $\mathcal{O}$ and a closed set F such that

$$F \subset E \subset \mathcal{O} \quad \text{and}$$

$$\mu(\mathcal{O} \setminus E) < \varepsilon \quad \text{and} \quad \mu(E \setminus F) < \varepsilon$$



A Borel measure μ satisfying (*) is called a Borel regular measure

Examples

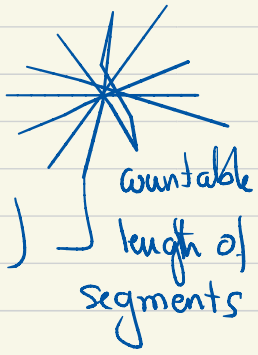
① hypothesis satisfied $(\mathbb{R}^n, m)$

② hypothesis not satisfied $(\mathbb{R}; \sum_{i=1}^{\infty} \delta_{\{x_i\}})$

$$\delta_{\{x_i\}}(E) = \begin{cases} 1 & \text{if } x_i \in E \\ 0 & \text{if } x_i \notin E \end{cases}$$

$x_i = i$ (hyp satisfied)

$x_i = \frac{1}{i}$ (hyp not satisfied)



$\mathbb{R} \supset E$ $\qquad \nu(E) = \int_E \frac{dx}{|x|} \qquad \nu(B(0,\varepsilon_0)) = +\infty$

How does one prove something like this proposition?

$$\mathcal{C} = \{ E \in \mathcal{B}_X : \forall \varepsilon > 0 \ \exists O_\varepsilon \text{ open } \& \ \exists F_\varepsilon \text{ closed s.t}$$

$$F_\varepsilon \subset E \subset O_\varepsilon, \ \mu(O_\varepsilon \setminus E) < \varepsilon \ \& \ \mu(E \setminus F_\varepsilon) < \varepsilon \}$$

(1) What do we want to show?

(2) How do we show that?

Goal: show $\mathcal{C} = \mathcal{B}_X$ $(\mathcal{C} \subset \mathcal{B}_X)$ $\boxed{\mathcal{B}_X \subset \mathcal{C}}$

(i) $\mathcal{C}$ is a σ-algebra

(ii) If F is closed $F \in \mathcal{C}$

To show that $\mathcal{B}_X = \mathcal{C}$ we need

Claim 1: $\mathcal{C}$ is a σ-algebra

Claim 2 $\mathcal{C}$ contains all open sets (thus it contains the σ-algebra generated by the open sets $\mathcal{B}_X$) - Hence $\mathcal{B}_X \subset \mathcal{C} \Rightarrow \mathcal{B}_X = \mathcal{C}$

$\mathcal{C} = \{ E \in \mathcal{B}_X : \forall \varepsilon > 0 \ \exists O_\varepsilon \text{ open } \& \ \exists F_\varepsilon \text{ closed s.t}$

(*) $\{ F_\varepsilon \subset E \subset O_\varepsilon , \ \mu(O_\varepsilon \setminus E) < \varepsilon \ \& \ \mu(E \setminus F_\varepsilon) < \varepsilon \}$

is a σ-algebra : (1) $X, \phi \in \mathcal{C}$

(2) $E \in \mathcal{C}$ (*) holds ; note $O_\varepsilon^c \subset E^c \subset F_\varepsilon^c$

$O_\varepsilon \setminus E = O_\varepsilon \cap E^c = E^c \cap (O_\varepsilon^c)^c = E^c \setminus O_\varepsilon^c$ $\quad O_\varepsilon^c$ closed

$E \setminus F_\varepsilon = E \cap F_\varepsilon^c = F_\varepsilon^c \cap (E^c)^c = F_\varepsilon^c \setminus E^c$ $\quad F_\varepsilon^c$ open

$\& \quad \mu(F_\varepsilon^c \setminus E^c) < \varepsilon$

$\& \quad \mu(E^c \setminus O_\varepsilon^c) < \varepsilon \implies E^c \in \mathcal{C}$

(3) $\{E_k\} \subset \mathcal{C}$ $\quad E = \bigcup_{k=1}^{\infty} E_k$ $\quad \forall \varepsilon > 0 \ \ \forall k \in \mathbb{N}$

$\exists \ \widetilde{F}_k \subset E_k \subset O_k$ s.t $\quad \mu(O_k \setminus E_k) < \varepsilon / 2^k$

$\widetilde{F}_k$ closed, O_k open

$\mu(E_k \setminus \widetilde{F}_k) < \varepsilon / 2^k$

$$\mathcal{O} = \bigcup_{k=1}^{\infty} O_k \quad \text{open} \quad E \subset \mathcal{O} \qquad \mathcal{O} \setminus E = \bigcup_{k=1}^{\infty} O_k \setminus \left(\bigcup_{k=1}^{\infty} E_k \right)$$

$$\mathcal{O} \setminus E = \bigcup_{k=1}^{\infty} O_k \cap \left(\bigcup_{k=1}^{\infty} E_k \right)^c = \bigcup_{k=1}^{\infty} O_k \cap \left(\bigcap_{j=1}^{\infty} E_j^c \right)$$

$$\subseteq \bigcup_{k=1}^{\infty} O_k \cap E_k^c = \bigcup_{k=1}^{\infty} O_k \setminus E_k$$

$$\mu(\mathcal{O} \setminus E) \leq \sum_{k=1}^{\infty} \mu(O_k \setminus E_k) \leq \varepsilon$$

let $F^* = \bigcup_{k=1}^{\infty} \widetilde{F}_k$ (not necessarily closed!) nevertheless

as before $\mu(E \setminus F^*) \leq \varepsilon$ (Note we have not used the hypothesis!)

let $F_k = \bigcup_{j=1}^{k} \widetilde{F}_j$ closed sets $F_k \subset F_{k+1}$

Claim: given $\varepsilon > 0$ $\exists F_\varepsilon$ closed s.t. $F_\varepsilon \subset F^*$ & $\mu(F^* \setminus F_\varepsilon) < \varepsilon$

$\mu(E \setminus F_\varepsilon) \leq \mu(E \setminus F^*) + \mu(F^* \setminus F_\varepsilon) \leq 2\varepsilon \rightsquigarrow$ $\mathcal{C}$ closed under countable unions