Definition: A collection of subsets $\mathcal{A} \subset 2^X$ is a σ-algebra provided

(i) $\emptyset, X \in \mathcal{A}$

(ii) $A \in \mathcal{A} \Rightarrow X - A \in \mathcal{A}$

(iii) $\{A_k\}_k \subset \mathcal{A} \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$

Remark: If $\mathcal{A}$ is a σ-algebra and $\{B_k\}_{k=1}^{\infty} \subset \mathcal{A}$ then

$$\bigcap_{k=1}^{\infty} B_k \in \mathcal{A}$$

Pf: $B_k \in \mathcal{A}$ $\overset{\text{(ii)}}{\Downarrow}$ $B_k^c \in \mathcal{A}$

$\overset{\text{(iii)}}{\rightsquigarrow}$ $\bigcup_k B_k^c \in \mathcal{A}$ $\overset{\text{(ii)}}{\rightsquigarrow}$ $\bigcap_{k=1}^{\infty} B_k \in \mathcal{A}$

$\bigcup_k B_k^c = \left(\bigcap_{k=1}^{\infty} B_k \right)^c$

Examples (1) 2^X (2) $\{\emptyset, X\}$

(3) μ-outer measure $\mathcal{A} = \{E \subset X, \mu\text{-measurable}\}$

(4)

Definition: A collection of subsets $\mathcal{M} \subset 2^X$ is a σ-algebra provided

(i) $\emptyset, X \in \mathcal{M}$

(ii) $A \in \mathcal{M} \Rightarrow X - A \in \mathcal{M}$

(iii) $\{A_k\} \subset \mathcal{M} \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{M}$

Theorem: If μ is an outer measure on a non-empty set X, then the collection of all μ-measurable subsets of X is a σ-algebra

Definitions: ① If $\mathcal{C} \subset 2^X$ non-empty collection of subsets of X, the σ-algebra generated by $\mathcal{C}$, denote $\sigma(\mathcal{C})$ is the smallest σ-algebra containing $\mathcal{C}$ $\sigma(\mathcal{C}) = \bigcap \mathcal{M}$ or $\mathcal{M}$ σ-algebra $\mathcal{C} \subset \mathcal{M}$

② The Borel σ-algebra of X (metric space) is the smallest σ-algebra of X containing the open sets of X. Notation $\mathcal{B}_X$

③ An outer measure μ is Borel if each Borel set is μ-measurable

Example in $\mathbb{R}^d$ $\quad R = [a_1, b_1] \times \dots \times [a_d, b_d]$ rectangle

volume of $R = (b_1 - a_1) \times \dots \times (b_d - a_d)$

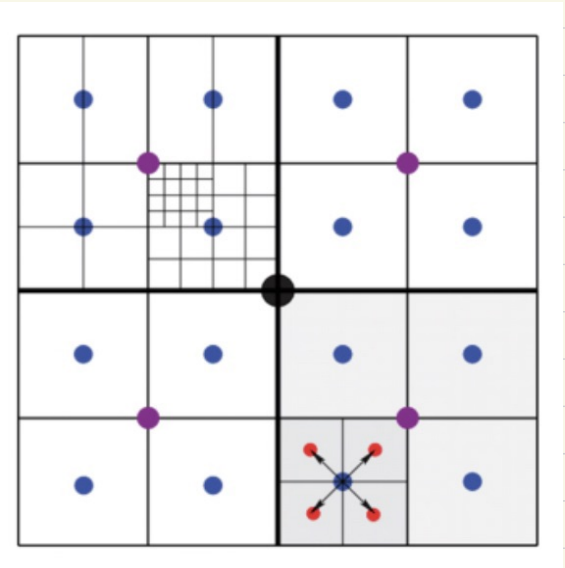
a cube is a rectangle such that $(b_i - a_i) = l \quad i = 1 \dots d$

its volume is l^d

Dyadic Cubes $\mathcal{D}$

$\mathcal{D}_0 = \{Q$ closed cubes with sidelength 1 & vertices integer coordinates$\}$

$\mathcal{D}_m = \{Q$ dyadic cube of sidelength 2^{-m} obtained by bisecting the original grid$\}$



$Q \in \mathcal{D}_m \quad l(Q) = 2^{-m}$ sidelength

$|Q| = 2^{-md}$

$$\mathcal{D} = \bigcup_{m \geq 0} \mathcal{D}_m$$

Theorem

Every open set in $\mathbb{R}^d$ can be written as a union of dyadic cubes

m_* assigns to every set in $\mathbb{R}^d$ a notion of size that coincides with our intuition.

1) Q cube $\quad m_*(Q) = |Q|$ ✓

2) Q_1, Q_2 cubes $\quad \ell(Q_1) = \ell(Q_2) \quad m_*(Q_1) = m_*(Q_2)$ ✓

3) $m_*(E + x) = m_*(E)$ translation inv. ✓
(rotation inv)

4) $A \cap B = \emptyset \quad m_*(A \cup B) = m_*(B) + m_*(B)$

↑
?

Definition: $E \subset \mathbb{R}^d$

$$m_* E = \inf \left\{ \sum_{j=1}^{\infty} |Q_j| : E \subset \bigcup_{j=1}^{\infty} Q_j \quad Q_j \text{ closed cubes} \right\}$$

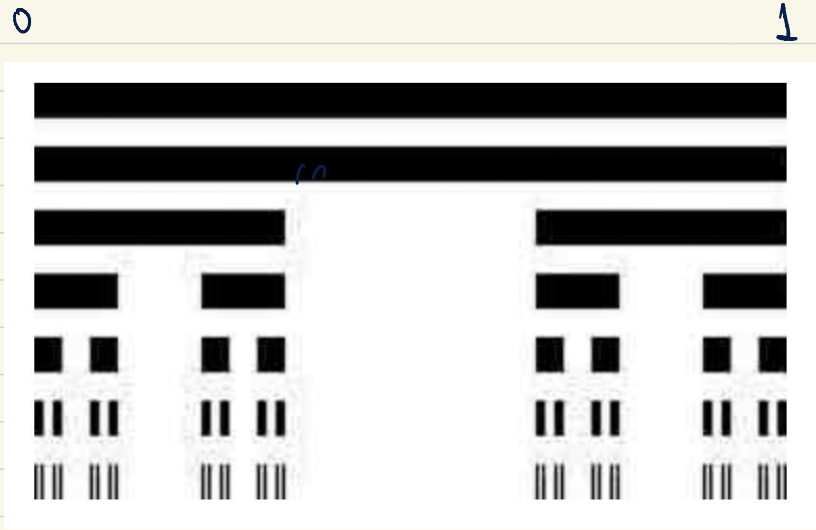
$$0 \leq m_* E \leq \infty$$

Examples ① $m_* \{x\} = 0$

② R rectangle $m_* (R) = |R| =$ volume

③ Cantor set $\mathcal{C}$

$\mathcal{C}_k$ = disjoint union of 2^k intervals of sidelength 3^{-k}



$$m_* (\mathcal{C}_k) \leq \left(\tfrac{2}{3}\right)^k \xrightarrow{k \to \infty} 0$$

$$m_* (\mathcal{C}) \leq m_* (\mathcal{C}_k) \xrightarrow{k \to \infty} 0$$

$$m_* (\mathcal{C}) = 0$$

$$\mathcal{C} = \bigcap_{k=0}^{\infty} \mathcal{C}_k$$

Remark: Given $E \subset \mathbb{R}^d$ $\forall \varepsilon > 0$ $\exists \{Q_i\}$ s.t $E \subset \bigcup Q_i$

$$\sum_{i=1}^{\infty} m_* (Q_i) \leq m_* (E) + \varepsilon$$

Lemma: m_* is an outer measure in $\mathbb{R}^d$, moreover

$$m_*(E + x) = m_*(E) \quad \text{(translation invariance)} \quad \forall x \in \mathbb{R}^d$$

Q: Is m_* additive in $(\mathbb{R}^d, 2^{\mathbb{R}^d})$; ie is it true that if $A, B \subset \mathbb{R}^d$ & $A \cap B = \emptyset$ then

$$m_*(A \cup B) \overset{?}{=} m_*(A) + m_*(B) \; ?$$

$$\overset{?}{=} m_*((A \cup B) \cap A) + m_*(A \cup B \setminus A)$$

A: we will see

Q: Are all sets in $\mathbb{R}^d$ m_*-measurable?

A: Consider the following equivalence relation in $[0,1]$

$$x \sim y \text{ iff } x - y \in \mathbb{Q}$$

let $\{E_\alpha\}_\alpha$ be the collection of equivalence classes then

$$[0,1] = \bigcup_\alpha E_\alpha \quad \text{disjoint union}$$

Consider the following equivalence relation in $[0,1]$

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Construct N by choosing exactly 1 element from each $E_\alpha \ni x_\alpha$

$$N = \{x_\alpha\}_\alpha$$

Theorem : The set N is not measurable

Pf $\{r_k\}_k = \mathbb{Q} \cap [-1,1]$ $\quad N_k = N + r_k$ ← disjoint

$$[0,1] \subset \bigcup_{k=1}^{\infty} N_k \subset [-1,2]$$

$x \in [0,1]$ $\exists\, x \in E_\alpha$ $\exists r_k$ s.t. $x - r_k = x_\alpha$ or $x_\alpha - r_k = x$

If N m_* measurable $m_*(N_k) = m_*(N)$

$$[0,1] \subset \bigcup_k N_k \subset [-1,2]$$

if N_k m_*-measurable since N_k disj

$$m_*[0,1] \leq m_*\left(\bigcup_k N_k\right) = \sum_k^\infty m_*(N_k)$$

↓ translation inv.

$$m_*[0,1] \leq \sum_k^\infty m_*(N) \leq m_*[-1,2]$$

$m_*(N) > 0$ $\qquad$ $m_*(N) = 0$

$*$

Definition : Let (X, ρ) be a metric space, an outer (exterior) measure μ_* on X is a metric outer (exterior) measure if

$$\mu_*(A \cup B) = \mu_*(A) + \mu_*(B)$$

whenever

$$d(A,B) = \inf_{\substack{a \in A \\ b \in B}} \rho(a,b) > 0$$

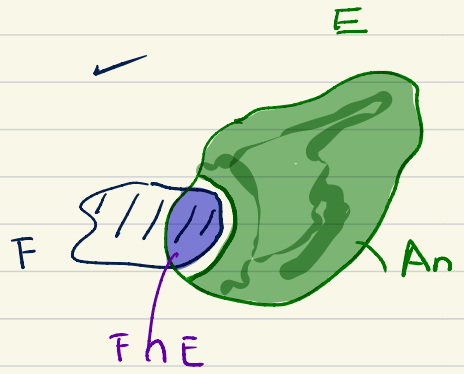
Homework : Show that m_* is a metric outer measure

Lemma : Let μ_* be a metric outer measure on (X, ρ), then all closed sets are μ_*-measurable, thus so are all open sets, and μ_* is Borel

Corollary : all closed and open sets in $\mathbb{R}^d$ are m_*-measurable and m_* is Borel.

Pf: Let $F \subset X$ closed set, $E \subset X$ if $\mu_* E = +\infty$

then $\mu_* E \geq \mu_* (F \cap E) + \mu_* (E \cap F^c)$ ✓



Assume $\mu_* E < \infty$ - For $n \in \mathbb{N}$ consider

$$A_n = \{ x \in F^c \cap E : d(x, F) \geq \frac{1}{n} \}$$

here $d(x, F) = \inf \{ \rho(x, y) : y \in F \}$

Claim 1: $A_n \subset A_{n+1}$ & $\bigcup_{n=1}^{\infty} A_n = F^c \cap E$ & $d(F \cap E ; A_n) \geq \frac{1}{n}$

Since μ_* is a metric outer measure, for any $n \in \mathbb{N}$

$$\mu_* E \geq \mu_* (E \cap F \cup A_n) \geq \mu_* (E \cap F) + \mu_* (A_n)$$

Claim 2: $\lim_{n \to \infty} \mu_* (A_n) = \mu_* (F^c \cap E)$