Theorem : Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of μ- measurable sets

i) The sets $\bigcup_{k=1}^{\infty} A_k$ and $\bigcap_{k=1}^{\infty} A_k$ are μ- measurable.

ii) If the sets $\{A_k\}_{k=1}^{\infty}$ are disjoint then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$$

iii) If $A_1 \subset A_2 \subset \dots A_k \subset A_{k+1} \dots$ then

$$\lim_{k \to \infty} \mu(A_k) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right)$$

iv) If $A_1 \supset \dots A_k \supset A_{k+1} \dots$ and $\mu(A_1) < \infty$ then

$$\lim_{k \to \infty} \mu(A_k) = \mu\left(\bigcap_{k=1}^{\infty} A_k\right)$$

A set $A \subseteq X$ is μ- measurable if for each set $B \subset X$

$$\boxed{\mu(B) = \mu(B \cap A) + \mu(B \setminus A)}$$

enough to show $\quad \mu(B) \geq \mu(B \cap A) + \mu(B \setminus A)$

Proof: i) For $n \geq 1$, $\bigcup_{k=1}^{n} A_k$ is μ- measurable : $B \subset X$

$$\mu(B) = \mu(B \cap A_1) + \mu(B \setminus A_1)$$

A_1 measurable

$$= \mu(B \cap A_1 \cap A_2) + \mu(B \cap A_1 \setminus A_2) + \mu((B \setminus A_1) \cap A_2)$$

A_2 is μ- measurable

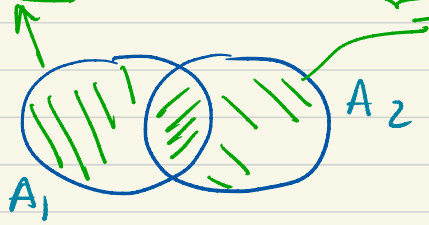
$B \cap A_1 \setminus A_2 = B \cap A_1 \cap A_2^c$

$B \cap A_1^c \cap A_2$

$$+ \mu((B \setminus A) \setminus A_2)$$

$B \cap A_1^c \cap A_2^c$

$$B \cap (A_1 \cup A_2) = B \cap A_1 \cap A_2 \cup B \cap A_1 \cap A_2^c \cup B \cap A_1^c \cap A_2$$



$A_1 \cup A_2$ μ- measurable

$$\mu(B) \geq \mu(B \cap (A_1 \cup A_2)) + \mu(B \setminus (A_1 \cup A_2))$$

$B \setminus (A_1 \cup A_2) = B \cap (A_1 \cup A_2)^c = B \cap A_1^c \cap A_2^c$

$$B \cap (A_1 \cup A_2) = B \cap (A_1 \cap A_2) \cup B \cap A_1^c \cap A_2 \cup B \cap A_1 \cap A_2^c$$

$$\mu(B \cap (A_1 \cup A_2)) \leq \mu(\;\downarrow\;) + \mu(\;\swarrow\;) + \mu(\;\swarrow\;)$$

$$\mu(B) \geq \mu(B \cap (A_1 \cup A_2)) + \mu(B \setminus (A_1 \cup A_2))$$

$\Rightarrow A_1 \cup A_2$ is μ-measurable $\Rightarrow \forall n \in \mathbb{N}$ | $\bigcup_{i=1}^{n} A_i$ is μ-measurable

---

A_i μ-meas. A_1^c is μ-meas $\Rightarrow \bigcup_{i=1}^{n} A_i^c$ is μ-meas

$\Rightarrow \left(\bigcup_{i=1}^{n} A_i^c\right)^c$ is μ-meas

$\left(\bigcup_{i=1}^{n} A_i^c\right)^c = \bigcap_{i=1}^{n} A_i$

Proof ii) $\{A_k\}_{k=1}^{\infty}$ μ- measurable disjoint sets

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$$

$$B_{j+1} = \bigcup_{k=1}^{j+1} A_k \qquad \mu\text{- measurable}$$

A_{j+1} is μ- measurable

$$\mu(B_{j+1}) = \mu(B_{j+1} \cap A_{j+1}) + \mu(B_{j+1} \cap A_{j+1}^c)$$

$$\mu(B_{j+1}) = \mu(A_{j+1}) + \mu(B_j) \underset{\text{repeat}}{=} \sum_{k=1}^{j+1} \mu(A_k)$$

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \geq \mu(B_{j+1}) \geq \sum_{k=1}^{j+1} \mu(A_k)$$

$\uparrow$ monotonicity

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \geq \sum_{k=1}^{j+1} \mu(A_k)$$

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \geq \sum_{j=1}^{\infty} \mu(A_k) \qquad j \to \infty$$

iii) If $A_1 \subset \dots \subset A_k \subset A_{k+1} \subset \dots$

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{j\to\infty} \mu(A_j)$$

if $\mu(A_{j_0}) = +\infty$ ✓

$$B_0 = A_1$$

$$B_j = A_{j+1} \setminus A_j$$

$$\bigcup_{j=1}^{\infty} B_j \cup A_1 = \bigcup_{j=1}^{\infty} A_j$$

$\{B_j\}_{j=0}^{\infty}$ disjoint

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \mu\left(\bigcup_{j=0}^{\infty} B_j\right) = \sum_{j=0}^{\infty} \mu(B_j)$$

$$= \mu(A_1) + \sum_{j=1}^{\infty} \mu(A_{j+1} \setminus A_j)$$

if $\mu(A_j) < \infty$ $\forall j$

A_j measurable

$A_j \subset A_{j+1}$

$$= \mu(A_1) + \sum_{j=1}^{\infty} \mu(A_{j+1}) - \mu(A_j)$$

$$= \lim_{j\to\infty} \mu(A_j)$$

iv) $A_1 \supset A_2 \supset \dots \quad A_k \supset A_{k+1} \dots \quad \mu(A_1) < \infty$

$$\lim_{k\to\infty} \mu(A_k) = \mu\left(\bigcap_{k=1}^{\infty} A_k\right)$$

Pf: $\underbrace{\mu(A_1) - \mu(A_k)}_{(l.h.s)_k} \qquad \underbrace{\mu(A_1) - \mu\left(\bigcap_{k=1}^{\infty} A_k\right)}_{(r.h.s)}$

$$\lim_{k\to\infty} (l.h.s)_k = \lim_{k\to\infty} (r.h.s) =$$

$\mu(A_1) - \mu(A_k) = \mu(A_1 \setminus A_k)$ bec A_j μ-meas

$\mu(A_k) \le \mu(A_1) < \infty \qquad A_1 \supset A_k$

$$\mu(A_1) - \mu(A_k) = \mu\left(\bigcup_{n=1}^{k} \underbrace{(A_1 \setminus A_n)}\right), \qquad A_1 \setminus A_n \subset A_1 \setminus A_{n+1}$$

$k \to \infty$

$$\mu(A_1) - \lim_{k\to\infty} \mu(A_k) = \underbrace{\mu\left(\bigcup_{n=1}^{\infty} A_1 \setminus A_n\right)}. \quad \text{by iii)}$$

$$A_1 \subset \bigcup_{n=1}^{\infty} A_1 \setminus A_n \cup \bigcap_{k=1}^{\infty} A_k$$

$$\mu(A_1) \le \mu\left(\bigcup_{n=1}^{\infty} A_1 \setminus A_n\right) + \overbrace{\mu\left(\bigcap_{k=1}^{\infty} A_k\right)}^{<\infty}.$$

$$\mu(A_1) - \lim_{k\to\infty} \mu(A_k) = \mu\left(\bigcup_{n}^{\infty} A_1 \setminus A_n\right)$$

$$\geq \mu(A_1) - \mu\left(\bigcap_{k=1}^{\infty} A_k\right)$$

$\mu(A_1) < \infty$

$$\Rightarrow \mu\left(\bigcap_{k=1}^{\infty} A_k\right) \geq \lim_{k\to\infty} \mu(A_k) \geq \mu\left(\bigcap_{k=1}^{\infty} A_k\right)$$

$$\bigcap_{k=1}^{\infty} A_k \subset A_k$$

is $\bigcup_{k=1}^{\infty} A_k$ & $\bigcup_{k=1}^{\infty} A_k$ μ-measurable.

$B_j = \bigcup_{k=1}^{j} A_k$ μ-measurable / to show $\bigcup_{k=1}^{\infty} A_k$ use def.

$B \subset X$ | $\overline{B_j \subset B_{j+1}}$ | $B \cap B_{j+1}^c \subset B \cap B_j^c$

$$\mu(B) = \mu(B \cap B_j) + \mu(B \setminus B_j)$$

$$= \mu \llcorner B(B_j) + \mu(B \setminus B_j)$$

$$\lim_{j \to \infty} \mu \llcorner B(B_j) = \mu \llcorner B\left(\bigcup_{j=1}^{\infty} B_j\right) = \mu \llcorner B\left(\bigcup_{k=1}^{\infty} A_k\right)$$

$\mu(B) < \infty$ $\quad \mu(B \cap B_1^c) < \infty$

$$\lim_{j \to} \mu(B \setminus B_j) = \mu\left(\bigcap_{j=1}^{\infty} B \setminus B_j\right) = \mu\left(\bigcap_{j=1}^{\infty} B \cap B_j^c\right)$$

$$= \mu\left(B \cap \bigcap_{j=1}^{\infty} B_j^c\right) = \mu\left(B \cap \left(\bigcup_{j=1}^{\infty} B_j\right)^c\right)$$

$$= \mu\left(B \setminus \bigcup_{k=1}^{\infty} A_k\right)$$