Theorem: (X, ρ) metric space, $E \subset X$ t.f.a.e.

i) E complete and totally bounded

ii) Bolzano - Weierstrass property, E is sequentially compact.

iii) Heine - Borel property, E is compact

Proof i) $\Leftrightarrow$ ii) : i) $\Rightarrow$ ii) & $\neg$ i) $\Rightarrow$ $\neg$ ii)

i) + ii) $\Rightarrow$ iii)

iii) $\Rightarrow$ ii) : $\neg$ ii) $\Rightarrow$ $\neg$ iii)

i) + ii) $\Longrightarrow$ iii) $E \subset \bigcup_{\alpha} O_\alpha$ O_α open find $\{O_{\alpha_i}\}_{i=1}^{m}$ s.t

$$E \subset \bigcup_{i=1}^{m} O_{\alpha_i}$$

Idea: show $\exists\, \varepsilon > 0$ s.t $B(x,\varepsilon) \cap E \neq \emptyset$ then $\exists \alpha$

$B(x,\varepsilon) \subset O_\alpha$ for some α.

Contradiction $\forall \varepsilon > 0$ $\exists x_\varepsilon$ (i.e. x_n) s.t $B(x_\varepsilon, \varepsilon) \cap E \neq \emptyset$ &

$\forall \alpha$ $B(x_\varepsilon, \varepsilon) \setminus O_\alpha = B(x_\varepsilon, \varepsilon) \cap O_\alpha^c \neq \emptyset$

(Derek)

$\varepsilon = \frac{1}{2^n}$

$\exists\, y_\varepsilon$ (i.e. y_n) $\in B(x_{\varepsilon_1}, \varepsilon) \cap E$ $B(y_\varepsilon, 2\varepsilon) \cap O_\alpha^c \neq \emptyset$

$B(x_\varepsilon, \varepsilon) \subset B(y_\varepsilon, 2\varepsilon)$

$\varepsilon = \frac{1}{n}$ $\exists\, y_n \in E$ s.t $B(y_n, \frac{2}{n}) \cap O_\alpha^c \neq \emptyset$ $\forall \alpha$

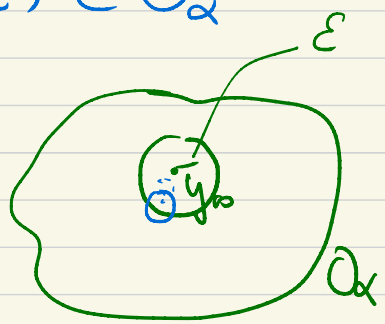
$\uparrow$ seq comp.

$\exists \{y_{n_k}\} \subset \{y_n\}$ $y_{n_k} \longrightarrow y_\infty \in E$

$y_{n_k} \to y_\infty \quad k \to \infty$

$E \rightsquigarrow \exists \alpha \mid y_\infty \in O_\alpha \mid$ open $\quad \exists \varepsilon > 0$

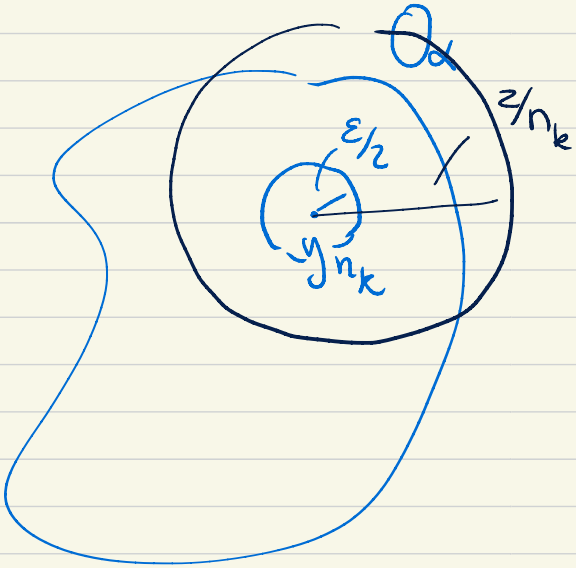
$B(y_\infty, \varepsilon) \subset O_\alpha$



$\exists k_0 \in \mathbb{N} \quad \forall k \geq k_0 \quad |y_{n_k} - y_\infty| < \varepsilon/2$

$B(y_{n_k}, \varepsilon/2) \subset B(y_\infty, \varepsilon) \subset O_\alpha$

$B(y_{n_k}, 2/n_k) \cap O_\alpha^c \neq \emptyset$



$\exists k_1 \in \mathbb{N} \quad 2/n_{k_1} < \varepsilon/2$

$k \geq k_1$

$B(y_{n_k}, \varepsilon/2) \cap O_\alpha^c \neq \emptyset$

Measure Theory

1 - Outer (exterior) measures

$X \neq \phi$ $\quad 2^X$ = collection of subsets of X

Definition: A mapping $\mu : 2^X \longrightarrow [0, \infty]$ is called an outer measure on X provided

(i) $\mu(\phi) = 0$

(ii) Subadditivity: if $A \subseteq \bigcup_{i=1}^{\infty} A_i$ then

$$\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$$



Remark: μ outer measure on X and $C \subset X$, then μ restricted to C; μLC is an outer measure defined by

$$\mu LC(A) := \mu(A \cap C) \qquad \forall A \subseteq X$$

Definition: A set $A \subseteq X$ is μ-measurable if for each set $B \subset X$

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A) = \mu(B \cap A) + \mu(B \cap A^c)$$

Remark: Since $B = B \cap A \cup B \setminus A$ by definition of outer measure

$$\mu(B) \leq \mu(B \cap A) + \mu(B \setminus A)$$

thus to show that a set A is μ-measurable it is enough to show

$$\mu(B) \geq \mu(B \cap A) + \mu(B \setminus A)$$

Theorem: Let μ be an outer measure on X.

i) If $A \subset B \subset X$ then $\mu(A) \leq \mu(B)$ (monotonicity)

ii) A is μ-measurable iff $X \setminus A$ is μ-measurable

iii) $\emptyset$ and X are μ-measurable - In general, if $\mu(A) = 0$ then A is μ-measurable

iv) if $C \subset X$, then each μ-measurable set is also $\mu \llcorner C$-measurable

Proof : i) definition of outer measure

iii) if $\mu(A) = 0$, since $A \cap B \subset A$ and $B \setminus A \subset B$

$\mu(A \cap B) \leq \mu(A) = 0$ & $\mu(B \setminus A) \leq \mu(B)$

iv) A μ- measurable, $B \subseteq X$

$$\mu_{LC}(B) = \mu(B \cap C) = \mu(B \cap C \cap A) + \mu(B \cap C \setminus A)$$

$$= \mu[(B \cap A) \cap C] + \mu[(B \setminus A) \cap C]$$

$$= \mu_{LC}(B \cap A) + \mu_{LC}(B \setminus A)$$

Theorem : Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of μ- measurable sets

i) The sets $\bigcup_{k=1}^{\infty} A_k$ and $\bigcap_{k=1}^{\infty} A_k$ are μ- measurable.

ii) If the sets $\{A_k\}_{k=1}^{\infty}$ are disjoint then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$$

iii) If $A_1 \subset A_2 \subset \dots A_k \subset A_{k+1} \dots$ then

$$\lim_{k \to \infty} \mu(A_k) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right)$$

iv) If $A_1 \supset \dots A_k \supset A_{k+1} \dots$ and $\mu(A_1) < \infty$ then

$$\lim_{k \to \infty} \mu(A_k) = \mu\left(\bigcap_{k=1}^{\infty} A_k\right)$$