2. Compact metric spaces in metric space (X, ρ)

1. (X, ρ) is a **complete** metric space if every Cauchy sequence converges

Complete

$E \subset \mathbb{R}^n$ closed

Not complete

$\mathbb{Q}$, $(0,1)$

2. $E \subset X$ is a **compact** set if every open cover of E has a finite subcover; ie if $\{O_\alpha\}$ family of open sets such that $E \subset \bigcup_\alpha O_\alpha$ then there exists $\{O_i\}_{i=1}^m \subset \{O_\alpha\}_\alpha$ such that

$$E \subset \bigcup_{i=1}^m O_i$$

Example: - closed & bounded set $\mathbb{R}^n$ (compact)

- $(0,1)$ not compact.

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3. a collection $\mathcal{F}$ of sets in X is said to have the finite intersection property if any finite subcollection of $\mathcal{F}$ has non-empty intersection.

Proposition : X is compact iff every collection $\mathcal{F}$ of closed sets with the finite intersection property has non-empty intersection.

Definitions (X, ρ) metric space

1- $E \subset X$ is **sequentially compact** if every sequence $\{x_n\} \subset E$ has a convergent subsequence

Examples:

- sequentially compact:
  - (compact $\Rightarrow$ sequentially compact)
- non-sequentially compact:
  - $\mathbb{R}$ $\{n\}_{n=1}^{\infty}$
  - $l^2 = \{(x_n)_n : \sum |x_n|^2 < \infty\}$ (unit sphere not seq.)

$$S = \{\underbrace{(x_n)_n}_{X} \in l^2 : \sum x_n^2 = 1\}$$

$$X_i = (\delta_{in})_n = (0 \dots 0, \underset{\text{ith position}}{\overset{\downarrow}{1}}, \dots 0)$$

$$\rho(x_i, x_j) = \|x_i - x_j\| = \sqrt{2}$$

(2) $E \subset X$ is **totally bounded** if for each $\varepsilon > 0$, E can be covered by finitely many balls of radius $\varepsilon > 0$; i.e. $\exists \{x_1, \dots x_n\} \subset X$ s.t.

$$E \subset \bigcup_{i=1}^{m} B(x_i, \varepsilon)$$

Example (X, ρ); $E \subset X$ bounded but not totally bounded

Example (X, ρ) ; $E \subset X$ bounded but not totally bounded.

E is bounded if there is $x_0 \in X$ and $R > 0$ such that

$E \subset B(x_0, R)$.

Theorem: (X, ρ) metric space, $E \subset X$ t.f.a.e.

i) E complete and totally bounded

ii) Bolzano - Weierstrass property, E is sequentially compact.

iii) Heine - Borel property, E is compact

i) $\Longleftrightarrow$ ii)

i) + ii) $\Longrightarrow$ iii)

iii) $\Longrightarrow$ ii)

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Proof i) $\Leftrightarrow$ ii) : i) $\Rightarrow$ ii) & $\neg$ i) $\Rightarrow \neg$ ii)

i) + ii) $\Rightarrow$ iii)

iii) $\Rightarrow$ ii) : $\neg$ ii) $\Rightarrow \neg$ iii)

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Proof i) $\Rightarrow$ ii) ① Given $\{x_n\} \subset E$ construct a Cauchy subsequence $\{x_{n_k}\} \subset E$

② Since E complete there is $x_\infty \in E$ s.t $\lim_{k \to \infty} x_{n_k} = x_\infty$

③ Claim: If $\{x_n\}$ Cauchy sequence and a subsequence $\{x_{n_k}\} \subset \{x_n\}$ converges to x_∞, then $\lim_{n \to \infty} x_n = x_\infty$

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Proof: $\neg i) \Rightarrow \neg ii)$

$\neg i)$ means that either: ① E is not complete

or

② E is not totally bounded

In both cases the goal is to construct a sequence that has NO convergent subsequences

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Proof: i) + ii) $\Rightarrow$ iii)

Let $E \subset \bigcup_{\alpha} O_\alpha$, O_α open, we need to show that there is a finite subcollection $\{O_{\alpha_i}\}_{i=1}^{k}$ such that $E \subset \bigcup_{i=1}^{k} O_{\alpha_i}$

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Idea: Show that there is $\varepsilon > 0$ so that if $B(x, \varepsilon) \cap E \neq \emptyset$ there exists α such that $B(x, \varepsilon) \subset O_\alpha$

Since E is totally bounded only finitely many such balls are needed to cover E, and thus only finitely many O_α's are needed to cover E.

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Proof iii) $\Rightarrow$ ii) iff $\neg$ ii) $\Rightarrow \neg$ iii)

$\neg$ ii) there is a sequence $\{x_n\}$ that does not have a convergent subsequence in E. Use it to construct a family of open balls that cover E but that does not admit a finite subcover (ie E is not compact)