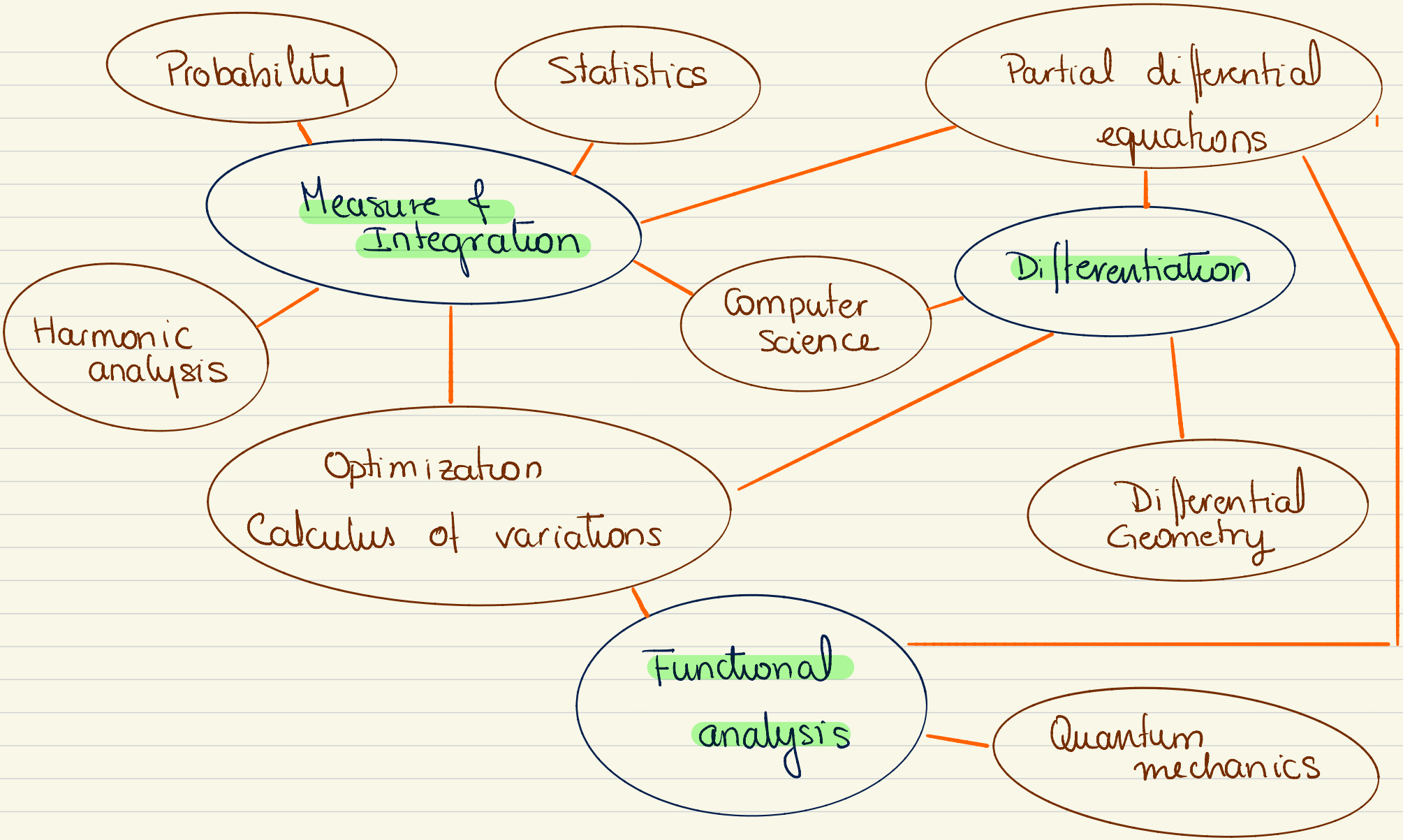

Probability
Statistics
Partial differential equations
Measure & Integration
Differentiation
Harmonic analysis
Computer Science
Optimization
Calculus of variations
Differential Geometry
Functional analysis
Quantum mechanics

Question : Is there another way to express the completeness of $\mathbb{R}$?

Cauchy criterion : Every Cauchy sequence in $\mathbb{R}$ converges.

least upper bound $\Rightarrow$ Cauchy criterion :

monotone convergence theo : monotone bounded seq converges

$\{x_n\} \subset \mathbb{R}$ Cauchy $\Rightarrow$ $\{x_n\}$ bounded

$a_n = \sup_{k \geq n} x_k$ $\quad a_{n+1} \leq a_n$ $\quad \lim_{n \to \infty} a_n = \inf_{k \geq 1} \sup_{n \geq k} x_n$

bounded $\quad = \limsup_{n \to \infty} x_n$

$b_n = \inf_{k \geq n} x_k$ $\quad b_{n+1} \geq b_n$ (choose monotonic subsequence

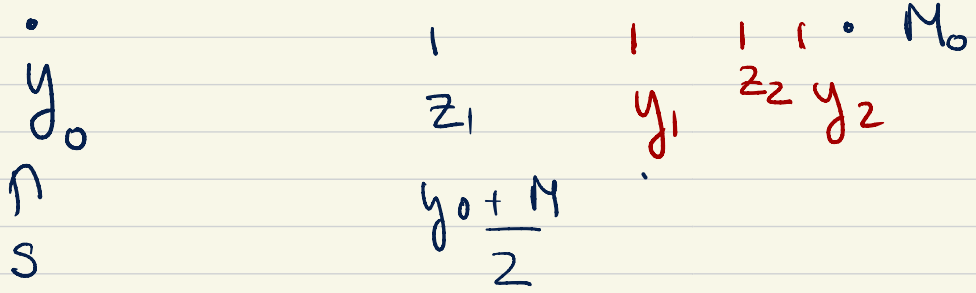
$\lim_{n \to \infty} b_n = \sup_{k \geq 1} \inf_{n \geq k} x_n$ it converges in $\mathbb{R}$)

If a Cauchy sequence has a convergent subsequence it converges. ($\limsup_{n \to \infty} x_n = \liminf_{n \to \infty} x_n = (\lim_{n \to \infty} x_n)$)

Cauchy criterion $\Rightarrow$ least upper bound (Jacob)

S bounded $\quad S \neq \emptyset \quad \forall x \in S \quad x \leq M_0 < \infty$

Construct $\{x_n\}$ Cauchy which conver



$\tilde{y}_m = \tilde{y}_{m-1} + \frac{\tilde{y}_{m-1} - \tilde{y}_{m-2}}{2}$

↑ if not upper bound for

$\tilde{y}_m \longrightarrow$ least upper bound of S

$|\tilde{y}_m - \tilde{y}_{m-1}| = \frac{1}{2}|\tilde{y}_{m-1} - \tilde{y}_{m-2}|$

if z_1 is an upper bound $\quad z_1 = M_1$

if z_1 is not an upper bound $\quad \exists\, y_1 \in S$

$z_2 = \frac{y_1 + M_0}{2}$ $\quad M_2$ if upper

$y_2 \in S$

$y_n \leq y_{n+1} \leq M_{n+1} \leq M_n$ they both converge to the least upper bound.

Question : Is there another way to express the completeness of $\mathbb{R}$?

Cauchy criterion : Every Cauchy sequence in $\mathbb{R}$ converges.

Proposition : The Cauchy criterion in $\mathbb{R}$ is equivalent to the least upper bound property in $\mathbb{R}$.

Def 1 : $\{x_n\} \subset \mathbb{R}$ is a Cauchy sequence if $\forall \varepsilon > 0 \quad \exists N \in \mathbb{N}$,
$\forall n, m \geq N \; ; \quad |x_n - x_m| < \varepsilon$

Def 2 : A sequence $\{x_n\} \subset \mathbb{R}$ converges to $l \in \mathbb{R}$ if $\forall \varepsilon > 0$
$\exists N \in \mathbb{N} \quad \forall n \geq N \quad |x_n - l| < \varepsilon \; ; \; \lim_{n \to \infty} x_n = l$

Metric Spaces

Definition: A metric space (X, ρ) is a non-empty set X with a function

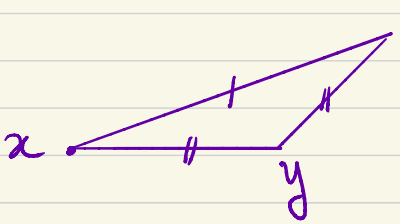
$$\rho : X \times X \longrightarrow \mathbb{R} \quad \text{a metric}$$

i) $\rho(x,y) \geq 0$

ii) $\rho(x,y) = 0 \iff x = y$

iii) $\rho(x,y) = \rho(y,x)$

iv) $\boxed{\rho(x,z) \leq \rho(x,y) + \rho(y,z)}$ Triangle inequality



Examples:

- $(\mathbb{R}^n, \text{euclidean})$ $\quad x = (x_1, \dots x_n)$

$$\|x-y\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \qquad \|x-y\|_p = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p}, \quad p \geq 1$$

- $(X, \text{discrete})$

$$\rho(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

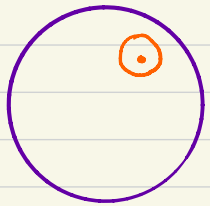
- $\mathcal{C}[0,1]$ continuous $\qquad \|f-g\|_\infty = \sup_{x \in [0,1]} |f(x) - g(x)|$

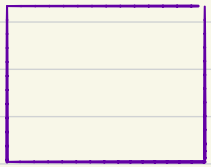
- $l^2 = \{(x_n)_{n=1}^{\infty} : x_n \in \mathbb{R}\} \qquad \|x-y\|_{l^2} = \left(\sum_{i=1}^{\infty} (x_i - y_i)^2\right)^{1/2}$

1. Open and closed sets in a metric space (X, ρ)

(Open) ball $B(x,r) = \{ y \in X : \rho(x,y) < r \}$ of center x and radius $r > 0$

Examples: $(\mathbb{R}^n, \text{euclidean})$ $(\mathbb{R}^n, \| \ \|_\infty)$ $\|x-y\|_\infty = \max_{i=1..n} |x_i - y_i|$





Definitions: ① $O \subset X$ open if for $\forall x \in O$ $\exists r > 0$ such that

$$B(x,r) \subset O$$

② $F \subset X$ closed if F^c open

Claim: the open ball is an open set

Proposition: 1) X, $\emptyset$ open sets

2) The union of any collections of open sets is open

3) The intersection of a FINITE collection of open sets is open

Definitions: $E \subset X$

① $\text{int}E = \bigcup_{\substack{O \subset E \\ O \text{ open}}} O$ interior of E = largest open set contained in E

i) $y \in \text{int } E$ iff $\exists r > 0$ $B(y,r) \subset E$

ii) E open iff $\text{int } E = E$

② $\overline{E} = \bigcap_{\substack{F \supset E \\ F \text{ closed}}} F$ closure of E = smallest closed set containing E

i) $y \in \overline{E}$ iff $\forall r > 0$ $B(y,r) \cap E \neq \emptyset$

ii) E is closed iff $\overline{E} = E$

Def: A metric space X is separable if it has a countable dense set D; ie $\overline{D} = X$

Examples: 1) separable $(\mathbb{R}^n, \| \ \|_2)$

2) Non-separable $(\mathbb{R}^n$, discrete metric$)$

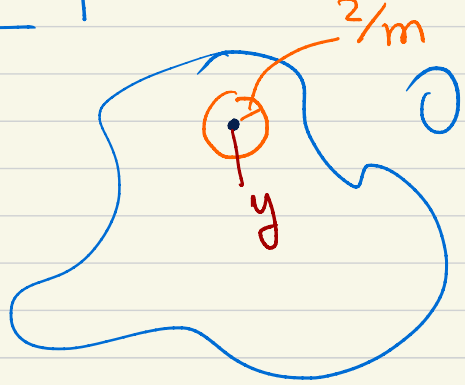
Proposition: A metric space (X, ρ) is separable iff there exists a countable family of open sets $\{O_i\}_i$ such that any open set $\mathcal{O} \subset X$

$$\mathcal{O} = \bigcup_{O_i \subset \mathcal{O}} O_i$$

X has the Lindelöf property

Pf: Vasili $\Rightarrow$

$D = \{x_n\}$ dense $\quad B_{nm} = B(x_n, \frac{1}{m})$ ← countable family of open sets



for m large enough $\exists x_n \in B(y; \frac{1}{2m})$ bec. D dense

Claim $y \in B(x_n, \frac{1}{m}) \subset \mathcal{O}$ triangle inequality

Leo $\Leftarrow$ $\{O_i\}_i$ choose $x_i \in O_i$ $D = \{x_i\}$ countable

to show $\overline{D} = X$, take $\mathcal{O}$ open $\neq \emptyset$ $\quad \mathcal{O} = \bigcup_{O_i \subset \mathcal{O}} O_i \quad \exists i$

$O_i \subset \mathcal{O} \Rightarrow x_i \in \mathcal{O}$.