HOMEWORK 1

TI problems due Wednesday October 7 at 10pm PT

Definition: Let (X, ρ) be a metric space. We say that $x \in X$ is a cluster point for $\{x_n\}_{n\geq 1} \subset X$ if $\forall\ \epsilon > 0$ and $\forall N \in \mathbb{N}$, $\exists n \in \mathbb{N}$, with $n \geq N$ such that $\rho(x, x_n) < \epsilon$.

Problem 1. Let (X, ρ) be a metric space, and let $\{x_n\}_{n\geq 1} \subset X$ be a Cauchy sequence. Assume that $\{x_n\}_{n\geq 1}$ has a cluster point $x \in X$. Prove that the sequence $\{x_n\}_{n\geq 1}$ converges to x.

Problem 2. Let $\{x_n\}_n \subset \mathbb{R}$, and $x \in \mathbb{R}$. Show that $x = \lim_{n\to\infty} x_n$ if and only if every subsequence of $\{x_n\}_{n\geq 1}$ has in turn a subsequence which converges to x.

Problem 3. Prove that $E \subset \mathbb{R}^n$ is compact if and only if E is closed and bounded.

Problem 4. Let (X, ρ) be a metric space. Prove that X is complete if and only if for any decreasing sequence $\{A_n\}$ of non-empty closed subsets of X (i.e.$\cdots \subset A_n \subset A_{n-1} \cdots \subset A_2 \subset A_1$, $A_i \neq \emptyset$, A_i closed) such that

$$\lim_{n\to\infty} \operatorname{diam} A_n = 0,$$

then

$$\cap_{n=1}^{\infty} A_n = \{x\} \quad \text{for some} \quad x \in X.$$

Recall

$$\operatorname{diam} A = \sup\{\rho(x, y) :\ x, y \in E\}.$$

Problem 5.

1. Let $g : \mathbb{R} \to [0, 1]$ be a non-decreasing function. Prove that the set of discontinuities of g is a countable.
2. Suppose that $f_n : \mathbb{R} \to [0, 1]$ for $n = 1, 2, \ldots$, and that each one of these functions is non-decreasing, that is, $f_n(x) \leq f_n(y)$ if $x \leq y$, for all n, x and y. Prove that there exist a function $g : \mathbb{R} \to [0, 1]$, a countable set $A \subset \mathbb{R}$, and a subsequence f_{n_k} such that $\lim_{k\to\infty} f_{n_k}(x) = g(x)$ for all $x \in \mathbb{R} \setminus A$.