Math 581A

Due November 29, 2017

Problem 1. Let μ be a Radon measure on $\mathbb{R}^n$. Assume that for $a \in \operatorname{spt}\mu = \Sigma$

$$1 \leq \limsup_{r\to 0} \frac{\mu(B(a,2r))}{\mu(B(a,r))} < \infty. \tag{1}$$

1. Show that for $\tau \geq 1$ and $a \in \Sigma$

$$1 \leq \limsup_{r\to 0} \frac{\mu(B(a,\tau r))}{\mu(B(a,r))} < \infty.$$

2. Prove that if there exist $\kappa > 1$ and $R > 0$ such that for $r \in (0,R)$ and all $a \in \Sigma$

$$\frac{\mu(B(a,2r))}{\mu(B(a,r))} \leq \kappa \tag{2}$$

then for any measure ν obtained as a weak limit of a sequence

$$(\mu(B(a,r_i)))^{-1} T_{a,r_i\#}\mu \text{ where } T_{a,r_i\#}\mu(E) = \mu(r_i E + a) \text{ for } E \subset \mathbb{R}^n \text{ Borel},$$

where $r_i \to 0$, the following statement holds:

$x \in \operatorname{spt}\nu$ if and only if there exists a sequence $x_i \in T_{a,r_i}(\Sigma)$ such that $x_i \to x$.

Problem 2. Let $E \subset \mathbb{R}^n$ satisfy $0 < \mathcal{H}^s(E) < \infty$, for $0 < s < 1$. Show that the density

$$\theta^s(E,x) = \lim_{r\to 0} \frac{\mathcal{H}^s(E \cap B(x,r))}{\omega_s r^s}$$

fails to exit at almost every point of E (i.e. $\theta^s(E,x)$ exists at most in a subset of E of $\mathcal{H}^s$ measure 0).

Remark: Marstrand proved this result in 1954. Later on he showed that if $s > 0$, and $\theta^s(E,x)$ exists on a subset $F \subset E$ with $\mathcal{H}^s(F) > 0$, then s must be an integer.

Problem 3. Let $\mu_j,\ \mu$ be Radon measures on a metric space X. Assume that for each $x \in X$, and each $j = 1, 2, ...$

$$\theta(\mu_j, x, r) = \frac{\mu_j(B_r(x))}{\omega_n r^n}, \text{ and } \theta(\mu, x, r) = \frac{\mu(B_r(x))}{\omega_n r^n},$$

are non-decreasing functions of r. Assume also that μ_j converges weakly to μ, and that $x_j \to x$ as $j \to \infty$. Prove that

$$\limsup_{j\to\infty} \theta(\mu_j, x_j) \leq \theta(\mu, x).$$

Here $\theta(\mu_j, x) = \lim_{r\to 0} \theta(\mu_j, x, r)$, and $\theta(\mu, x) = \lim_{r\to 0} \theta(\mu, x, r)$.

Remark: Note that in particular if $\mu_j = \mu$ for each j and $\theta(\mu, x, r)$ is a non-decreasing function of r, then the result above proves the upper semi-continuity of the density.

Problem 4. Let $M \subset \mathbb{R}^m$, $0 < n < m$, and $\mu = \mathcal{H}^n \llcorner M$. Assume that μ is a Radon measure, and that for each $x \in M$ $\theta(\mu, x, r) = \frac{\mu(B_r(x))}{\omega_n r^n}$ is a non-decreasing function of r. Let $\lambda_j > 0$ be a sequence converging to 0 as $j \to \infty$. For $x \in M$, let

$$M_j = \frac{1}{\lambda_j}(M - x) = \{y = \frac{1}{\lambda_j}(z - x) : z \in M\},$$

and

$$\mu_j = \mathcal{H}^n \llcorner (M_j \cap B_1(0)).$$

Show that for each j, μ_j is a Radon measure. Prove that there exists a subsequence μ_{j_k} of μ_j that converges weakly to a Radon measure ν, and that

$$(*) \qquad \theta(\mu, x) = \theta(\nu, 0).$$

Note that in particular $(*)$ asserts that $\lim_{r\to 0} \theta(\nu, 0, r)$ exits.

Remark: The situation described in Problem 4 occurs when M is a minimal n-dimensional submanifold of $\mathbb{R}^m$. In that case $\nu = \mathcal{H}^n \llcorner C$, where C is a cone of vertex 0. C is a tangent cone of M at x. As defined this cone depends on the subsequence λ_{j_k}. One of the big open questions in the subject is whether there is a unique tangent cone. Moreover the set $\{x \in M : \theta(\mu, x) = 1\}$ is open and smooth. The set $\{x \in M : \theta(\mu, x) > 1\}$ is a closed set of Hausdorff dimension at most $n - 1$.