Math 581A

Due October 25, 2017

Problem 1. A Radon measure μ on $\mathbb{R}^n$ is said to be doubling, if there exists a constant $C = C(n)$ depending only on n, such that for every $r > 0$ and every $x \in \mathbb{R}^n$

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

Assume that $\mu(B(0, 1)) < \infty$. Show that for any open set $U \subset \mathbb{R}^n$, and $\delta > 0$, there exists a countable collection $\mathcal{G}$ of disjoint closed balls in U such that $\operatorname{diam} \mathrm{B} \leq \delta$ for all $B \in \mathcal{G}$, and

$$\mu(U \setminus \bigcup_{B \in \mathcal{G}} B) = 0.$$

Problem 2. Let μ be a Radon measure in $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, let

$$M_\mu f(x) = \sup_{r>0} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} |f| \, d\mu,$$

if f is a μ-measurable function, and

$$M_\mu \nu(x) = \sup_{r>0} \frac{\nu(B(x, r))}{\mu(B(x, r))},$$

if ν is a Radon measure in $\mathbb{R}^n$.

- Show that there exists a constant $C < \infty$ depending only on n, with the following property: if μ and ν are Radon measures in $\mathbb{R}^n$, then

$$\mu\left(\{x \in \mathbb{R}^n : M_\mu \nu(x) > t\}\right) \leq Ct^{-1}\nu(\mathbb{R}^n).$$

- Show that for $1 < p < \infty$ there exists a constant $C_p < \infty$, depending only on n and p with the following property: if μ is a Radon measure in $\mathbb{R}^n$, then

$$\int (M_\mu f)^p \, d\mu \leq C_p \int |f|^p \, d\mu,$$

for all μ-measurable functions f.