Rectifiability of sets and measures

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State of affairs I

Let μ be a Radon measure in $\mathbb{R}^m$ such that

$$0 < \theta^n(\mu, a) = \lim_{r \to 0} \frac{\mu(B(a,r))}{r^n} < \infty \text{ for } \mu - a.e\ a \in \mathbb{R}^m$$

Then for μ-a.e. $a \in \mathbb{R}^m$:

- $\mathcal{F} \subset \mathcal{M}$, where $\mathcal{F}$ is the set of n-flat measures in $\mathbb{R}^m$ and $\mathcal{M}$ is the set of n-uniform measures in $\mathbb{R}^m$ with 0 in their support.
- $\text{Tan}(\mu, a) \subset \mathcal{M}$ and $\text{Tan}(\mu, a) \cap \mathcal{F} \neq \emptyset$. .
- If $(*)$ holds then $\text{Tan}(\mu, a) \subset \mathcal{F} \implies \mu$ n-rectifiable..

$$(*) \qquad d_r(\Phi, \mathcal{F}) < \epsilon_0 \quad \forall r \geq r_0 > 0 \text{ then } \Phi \in \mathcal{F}.$$

State of affairs II

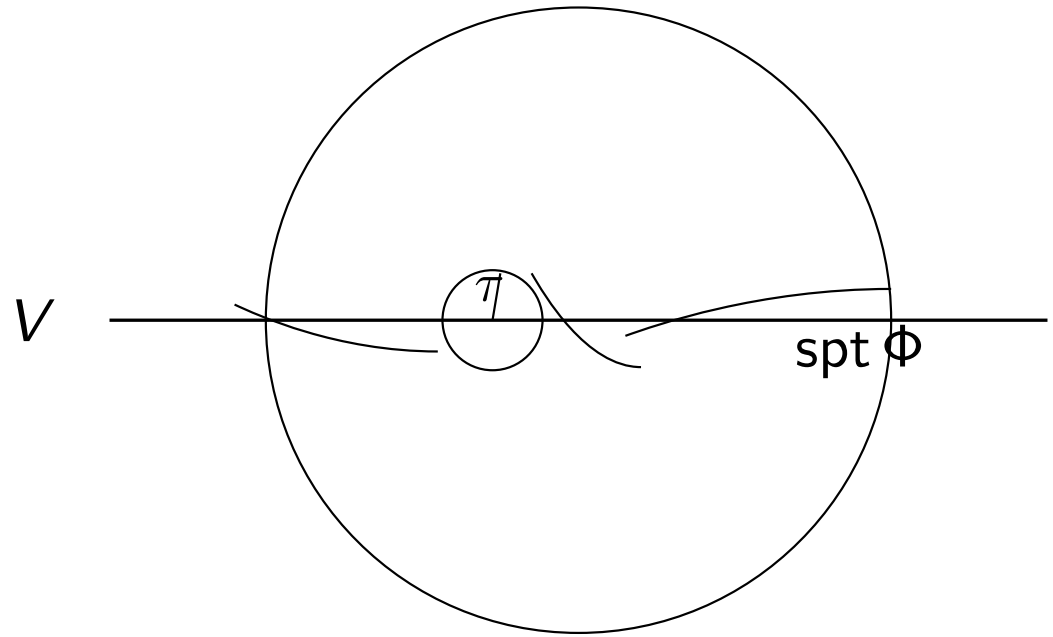
- If $d_r(\Phi, \mathcal{F}) < \epsilon_0$ for $r \geq r_0 > 0$ there is V_r such that

$$(\spadesuit) \qquad \frac{1}{r^{n+2}} \int_{B_r} \text{dist}^{\,2}(x, V_r)\, d\Phi \leq \epsilon_1.$$

- $(\spadesuit)$ implies $\sup_{y \in \text{spt}\,\Phi \cap B_r(1-2\epsilon_1^{1/2n})} d(y, V_r) \leq \epsilon_2 r.$

- Here $\epsilon_2 = \epsilon_1^{1/2n}$ and $\epsilon_1 = C(n)\epsilon_0$.

Question to address



Can there be holes of size τ if $\int_{B_1} \text{dist}^2(x, V)\, d\Phi < \epsilon_1$?

NO if ϵ_1 is small enough.

(♠) yields additional geometric information for Φ n-uniform

Lemma: Given $\tau > 0$ there exists $\epsilon_1 = \epsilon_1(\tau, n, m) > 0$ such that if Φ is n-uniform in $\mathbb{R}^m$ with $0 \in \operatorname{spt} \Phi$, $\Phi(B_1) = 1$ and for some $V \in G(m, n)$

$$\int_{B_1} \operatorname{dist}^2(x, V)\, d\Phi < \epsilon_1,$$

then for all $z \in V \cap B_1$

$$B(z, \tau) \cap \operatorname{spt} \Phi \neq \emptyset.$$

Proof by contradiction

Assume there is $\tau > 0$ s.t. for all $k \in \mathbb{N}$ there are Φ_ks n-uniform in $\mathbb{R}^m$ with $0 \in \operatorname{spt} \Phi_k$, $\Phi_k(B_1) = 1$, $V_k \in G(m, n)$

$$\int_{B_1} \operatorname{dist}^2(x, V_k)\, d\Phi_k < \frac{1}{2^k},$$

and $z_k \in V_k \cap B_1$ such that

$$B(z_k, \tau) \cap \operatorname{spt} \Phi_k = \emptyset.$$

- $\Phi_k(B(0, r)) = r^n$ for all $r > 0$ thus there exists a subsequence of the Φ_k which converges weakly to Φ_∞ also n-uniform. $(*)$
- $G(m, n)$ is compact there is a subsequence of the V_k converges to $V_\infty \in G(m, n)$ in the Hausdorff distance sense uniformly in compact sets.
- $\overline{B_1}$ is compact there is a subsequence of the z_k converges to $z_\infty \in \overline{B_1} \cap V_\infty$.

- $\Phi_k \rightharpoonup \Phi_\infty$, $\operatorname{spt}\Phi_k \to \operatorname{spt}\Phi_\infty$ $(*)$, $V_k \to V_\infty$, and $z_k \to z_\infty$.
- For $y \in \operatorname{spt}\Phi_k \cap B_k$ where $B_{1-2.2^{-k/2n}} = B_k$, $\operatorname{dist}(y, V_k) \leq 2^{-k/2n}$.
- For k large enough if $y \in \operatorname{spt}\Phi_k \cap B_k$ then $\operatorname{dist}(y, V_\infty) \leq 4.2^{-k/2n}$.
- Let $\pi : \mathbb{R}^m \to V_\infty$, $\Psi_k = \pi[\Phi_k \llcorner B_k]$. Note that $\operatorname{spt}\Psi_k \subset V_\infty$. For $f \in \mathcal{L}(1)$

$$\begin{aligned}\left|\int f\, d\Phi_k - \int f\, d\Psi_k\right| &\leq \left|\int_{B_k} |f(z) - f(\pi(z))|\, d\Phi_k + \int_{B_1 \setminus B_k} |f|\, d\Phi_k\right| \\ &\leq 4.2^{-k/2n} + \Phi_k(B_1 \setminus B_k) \leq C 2^{-k/2n}.\end{aligned}$$

- W.l.o.g we may assume that $\Psi_k \rightharpoonup \Psi_\infty$.
- Since $F_1(\Phi_k, \Psi_k) \to 0$ as $k \to \infty$, $F_1(\Phi_\infty, \Psi_\infty) = 0$, $\Psi_\infty = \Phi_\infty \llcorner B_1$ is locally n-uniform thus $\Psi_\infty = \mathcal{H}^n \llcorner (V_\infty \cap B_1) = \Phi_\infty \llcorner B_1$.

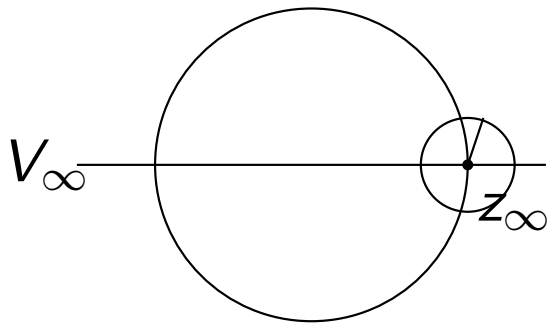
Contradiction

- Since for $z_k \in V_k \cap B_1$

$$B(z_k, \tau) \cap \operatorname{spt} \Phi_k = \emptyset,$$

$$B(z_\infty, \tau) \cap \operatorname{spt} \Phi_\infty = \emptyset.$$

Since $z_\infty \in V_\infty \cap \overline{B_1}$ and $\operatorname{spt} \Phi_\infty \cap B_1 = V_\infty \cap B_1$ then $B(z_\infty, \tau) \cap \operatorname{spt} \Phi_\infty \neq \emptyset$ which is a contradiction.



L^2 flatness implies bilateral L^∞ flatness for the support of an n-uniform in $\mathbb{R}^m$

Corollary: Given $\tau > 0$ there exists $\epsilon_1 = \epsilon_1(\tau, n, m) > 0$ such that if Φ is n-uniform in $\mathbb{R}^m$ with $0 \in \operatorname{spt}\Phi$, $\Phi(B_1) = 1$ and for some $V_r \in G(m, n)$

$$\frac{1}{r^{n+2}} \int_{B_r} \operatorname{dist}^2(x, V_r)\, d\Phi < \epsilon_1,$$

$$\frac{1}{r} D[V_r \cap B_r, \operatorname{spt}\Phi \cap B_r] \leq \tau + 2\epsilon_1^{1/2n}.$$

D denotes the Hausdorff distance.

Disappointment is appropriate! Our goal is to show that $\operatorname{spt}\Phi \in G(m, n)$ with huge amount of work we have shown that it is close to flat!

Tangent measure at infinity

Lemma: Let Φ be n-uniform in $\mathbb{R}^m$ there exists a *unique* Radon measure λ such that for all $x \in \mathbb{R}^m$

$$r^{-n} T_{x,r}[\Phi] \rightharpoonup \lambda \text{ as } r \to \infty.$$

λ is n-uniform and for $t > 0$, $\lambda(tA) = t^n \lambda(A)$, i.e. the support of λ is a cone.

Definition: Φ n-uniform in $\mathbb{R}^m$ is flat at infinity if

$$\text{Tan}(\Phi, \infty) = \{c\lambda : c > 0\} \subset \mathcal{F}.$$

Remark: λ is independent of the sequence. To show this Preiss used all the moments of a measure.

(♠) yields information about the tangent measure at infinity

Let Φ be n-uniform in $\mathbb{R}^m$ with $0 \in \operatorname{spt}\Phi$, $\Phi(B_1) = 1$ such that for $r \geq r_0 > 0$ there is $V_r \in G(m,n)$ so that

$$\frac{1}{r^{n+2}} \int_{B_r} \operatorname{dist}^2(x, V_r)\, d\Phi < \epsilon_1,$$

then for $r \geq r_0 > 0$

$$\int_{B_1} \operatorname{dist}^2(x, V_r)\, d(r^{-n} T_{0,r}[\Phi]) < \epsilon_1.$$

Using the compactness of $G(m,n)$ we know that there exists a sequence $r_i \to \infty$ to that $V_{r_i} \to V \in G(m,n)$. Moreover $r_i^{-n} T_{0,r_i}[\Phi] \rightharpoonup \lambda$ where $\lambda \in \operatorname{Tan}(\Phi, \infty)$ satisfies

$$\int_{B_1} \operatorname{dist}^2(x, V)\, d\lambda \leq \epsilon_1.$$

Flatness at infinity

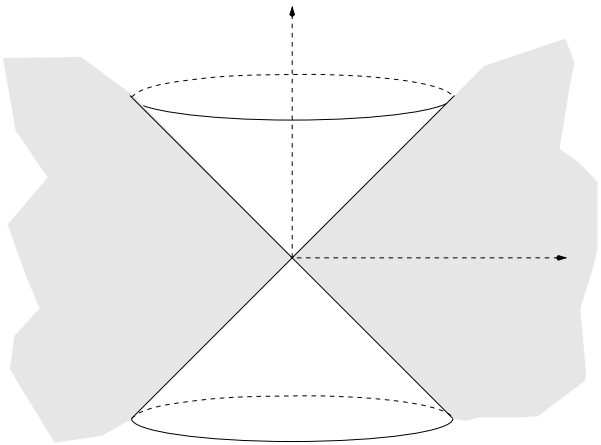
Theorem (Preiss) There exists $\epsilon = \epsilon(m, n) > 0$ such that if Φ is n-uniform in $\mathbb{R}^m$, then

- if $n = 1, 2$ then Φ is flat,
- if $n \geq 3$, $\lambda = [\Phi(B_1)]^{-1} \lim_{r \to \infty} r^{-n} T_{0,r}[\Phi]$ and

$$\int_{B_1} \text{dist}^2(x, V)\, d\lambda \leq \epsilon,$$

then Φ is flat at infinity, i.e. $\lambda \in \mathcal{F}$.

Recall in $\mathbb{R}^{n+1}$ there is an n-uniform measure which is not flat at infinity



Kowalski-Preiss

Theorem Let Φ is n-uniform in $\mathbb{R}^m$, such that $0 \in \operatorname{spt}\Phi$. Assume

$$\Phi(B_r(x)) = \omega_n r^n \qquad \forall x \in \operatorname{spt}\Phi, \quad \forall r > 0.$$

Then there is a degree 2 polynomial P such that

$$\operatorname{spt}\Phi \subset \{x : P(x) = 0\}.$$

Moments of an n-uniform measure

For $r > 0$ define

$$b_r = \frac{n+2}{2\omega_n r^{n+2}} \int_{B_r} (r^2 - |y|^2) y \, d\Phi(y), \qquad \text{(first moment)}$$

$$Q_r(x) = \frac{n+2}{\omega_n r^{n+2}} \int_{B_r} \langle x, y \rangle^2 \, d\Phi(y) \quad \text{for} \quad x \in \mathbb{R}^m. \qquad \text{(second moment)}$$

Note that Q_r is a non-negative symmetric quadratic form,

$$Tr(Q_r) = \frac{n+2}{\omega_n r^{n+2}} \int_{B_r} |y|^2 d\Phi(y) = n \qquad (*).$$

Applications of the moments - Uniform measures and tangent at infinity

A calculation shows that for $x \in \operatorname{spt}\Phi \cap B_{r/2}$

$$\left|2\langle b_r, x\rangle + Q_r(x) - |x|^2\right| \leq C\frac{|x|^3}{r}$$

There are sequences $r_i \to \infty$ such that $b_{r_i} \to b$ in $\mathbb{R}^m$, $Q_{r_i} \to Q$ a non-negative symmetric quadratic form with $Tr(Q) = n$. Moreover

$$\operatorname{spt}\Phi \subset \{x : 2\langle b, x\rangle + Q(x) - |x|^2 = 0\}.$$

Let $\lambda = \lim_{r\to\infty} r^{-n} T_{0,r}[\Phi]$. If $x \in \operatorname{spt}\Phi$ then $\frac{x}{r} \in \operatorname{spt} r^{-n} T_{0,r}[\Phi]$. For $z \in \operatorname{spt}\lambda$, there exists $r_i \to \infty$ and $x_i \in \operatorname{spt}\Phi$ so that $z = \lim_{i\to\infty} \frac{x_i}{r_i}$. Thus

$$2\frac{1}{r_i}\langle b, \frac{x_i}{r_i}\rangle + Q(\frac{x_i}{r_i}) - |\frac{x_i}{r_i}|^2 = 0 \implies Q(z) - |z|^2 = 0.$$

$$\operatorname{spt}\lambda \subset \{x : Q(x) - |x|^2 = 0\}.$$

Proof of the theorem

Assume that

$$\int_{B_1} \text{dist}^{\,2}(x, V)\, d\lambda \leq \epsilon,$$

We can find an orthonormal basis for $\mathbb{R}^m$ where Q is diagonal. Let $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m \geq 0$ with eigenvectors $\{e_i\}_{1 \leq i \leq m}$ and

$$\int_{B_1} \text{dist}^{\,2}(x, \text{span}_{1 \leq k \leq m-n}\{e_{n+k}\})\, d\lambda \leq \epsilon \implies$$

$$\int \sum_{k=1}^{m-n} x_{n+k}^2\, d\lambda = \omega_n \sum_{k=1}^{m-n} \alpha_{n+k} < \epsilon.$$

Recall $\sum_1^m \alpha_i = n$. If $\alpha_n < 1$ then

$$\begin{aligned} \alpha_i - 1 &\leq (n-1)(1-\alpha_n) \qquad \text{for } i = 1, \cdots, n-1 \\ \alpha_j &\leq \alpha_n < 1 \qquad \text{for } n \leq j \leq m. \qquad (*) \end{aligned}$$

For $\tau > 0$ to be chosen if ϵ is small enough then there is $x \in \operatorname{spt}\lambda \cap B_1$ such that $\|x - e_n\| < \tau$. This implies

$$\sum_{i=1}^{n-1} x_i^2 + (1 - x_n)^2 + \sum_{j=n+1}^{m} x_j^2 < \tau^2$$

$$\begin{aligned}
0 &= Q(x) - |x|^2 = \sum_{i=1}^{m} (\alpha_i - 1) x_i^2 \leq \sum_{i=1}^{n} (\alpha_i - 1) x_i^2 \\
&\leq (n-1)(1-\alpha_n) \sum_{i=1}^{n-1} x_i^2 + (\alpha_n - 1) x_n^2 \\
&\leq (n-1)(1-\alpha_n)\tau^2 - (1-\alpha_n)(1-\tau)^2 \\
&= (1-\alpha_n)((n-2)\tau^2 - 1 + @\tau) < 0 \qquad \text{for } \tau \text{ small enough.}
\end{aligned}$$

Thus $\alpha_n = 1$, since $\alpha_i \geq \alpha_n$ for $i \leq n$ and $\sum_1^m \alpha_i = n$ then $\alpha_i = 1$ for $1 \leq i \leq n$ and $\alpha_j = 0$ for $n+1 \leq j \leq m-n$. For $x \in \operatorname{spt} \lambda$,

$$Q(x) - |x|^2 = -\sum_{i=n+1}^{m} x_i^2 = 0 \implies \lambda \in \mathcal{F}.$$

Remark: To show that if Φ is n-uniform and for $r \geq r_0 > 0$ there is $V_r \in G(m,n)$ so that

$$\frac{1}{r^{n+2}} \int_{B_r} \operatorname{dist}^2(x, V_r)\, d\Phi < \epsilon_1,$$

then Φ is flat we need to use higher order moments. The goal is to show that if Φ is flat at infinity then the term $\langle b, x \rangle = 0$ for all $x \in \operatorname{spt} \Phi$.

Does the density ratio encode further smoothness of the support?

- Let $\theta^n(x,r) = \frac{\mu(B(x,r))}{\omega_n r^n}$ denote the density ratio of μ for $x \in \mathbb{R}^m$ and $r > 0$.

- If $\operatorname{spt}\mu = \Sigma$ and

$$\left| \frac{\mu(B(x,r))}{\omega_n r^n} - 1 \right| \leq C r^\alpha \text{ for } x \in \Sigma \text{ and } 0 < r \leq 1$$

  then $\lim_{r\to 0} \theta^n(x,r) = 1$ and Σ is rectifiable (Preiss).

- If Σ is a $C^{1,\gamma}$ n-dimensional submanifold of $\mathbb{R}^m$ then

$$\left| \frac{\mathcal{H}^n(B(x,r)\cap\Sigma)}{\omega_n r^n} - 1 \right| \leq C r^{2\gamma} \text{ for } x \in \Sigma \text{ and } 0 < r \leq R.$$

Hölder convergence of the density ratio implies further regularity

Theorem For $\alpha > 0$ there exists $\gamma = \gamma(\alpha, n, m) > 0$ such that if $\Sigma = \operatorname{spt} \mu \subset \mathbb{R}^m$, satisfies

$$|\theta^n(x, r) - 1| \leq C r^\alpha \text{ for } x \in \Sigma \text{ and } 0 < r \leq 1,$$

then

- if $n = 1, 2$ then Σ is a $C^{1,\gamma}$ n-dimensional submanifold of $\mathbb{R}^m$.
- if $n \geq 3$, $\Sigma = \mathcal{R} \cup \mathcal{S}$, where $\mathcal{R}$ is relatively open $C^{1,\gamma}$ n-dimensional submanifold of $\mathbb{R}^m$ and $\mathcal{S}$ is a closed set with $\mathcal{H}^n(\mathcal{S}) = 0$ Moreover if $m = n + 1$ then $\dim \mathcal{S} \leq n - 3$.

Contributions:

- David-Kenig-Toro
- Preiss-Tolsa-Toro

Global results

Theorem (Lewis) For $\alpha > 0$ there exists $\gamma = \gamma(\alpha) > 0$ such that if $\Sigma = \operatorname{spt} \mu \subset \mathbb{R}^4$, satisfies

$$\left| \frac{\mu(B(x,r))}{\omega_3 r^3} - 1 \right| \leq C r^{\alpha} \text{ for } x \in \Sigma \text{ and } 0 < r \leq 1,$$

then locally Σ is a $C^{1,\gamma}$ image via a map in $\mathbb{R}^4$ of either the unit disc in $\mathbb{R}^3$ or a neigborhood of the origin in the Kowalski-Preiss cone.

Open questions and Work in progress

- Size of the singular set in general (Nimer).
- Structure of the singular set.
- Classification of n-uniform measures in $\mathbb{R}^m$.
- Use of Preiss' ideas to study the structure of the harmonic measure in the absence of a density (Kenig -Preiss-Toro; Azzam-Mourgoglou-Tolsa).
- Use of Preiss' ideas to study different families of tangent objects (Badger, Engelstein, Lewis, Toro).